LECTURE 1 – STRUCTURAL ENGINEERING II Yaoundé, 17/11/2014

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REVIEW OF STATICS

- Structures are considered as a system of rigid bodies, connected by internal restraints and connected to the exterior by external restraints.
- Restraints (and supports) are considered as: { frictionless / with a "double" effect

Friction, wich exists in practice, is always against a possible displacement, so if you disregard it, you are considering a situation that is worse than the actual one → it is "on the safe side".

Also unidirectional restraints are supposed to work in both directions

Example :  this roller can apply a reaction in both the vertical directions 

In practice, you should check this assumption!

From a mechanical point of view, restraints exert forces; from a kinematic point of view, restraints block displacements. Forces and displacements are defined in a wide sense, including moments and rotations

- By assumption, structural members are considered, in this phase, as rigid bodies, so they do not deform and it is possible to solve the problem of equilibrium with reference to the initial configuration of the structure.

This is acceptable in most cases, but not in all the situations: there are certain problems ("special" ones) where the effect of deformations must be included in the analysis, for example in the case of instability (buckling failures).

- The general procedure to solve the problem of equilibrium consists in using a design sketch of the structure, with proper forces that represent the external restraints, and then find the values of those forces (i.e. the reaction) by means of equilibrium equations. Then, you can solve the problem of finding the internal forces along the members of the structure.

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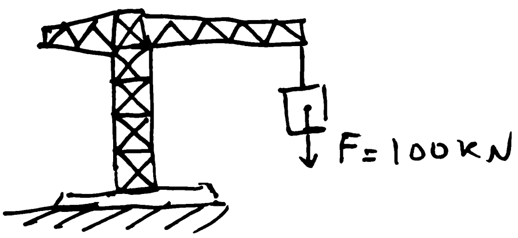
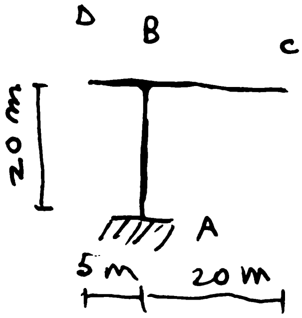
EXAMPLE

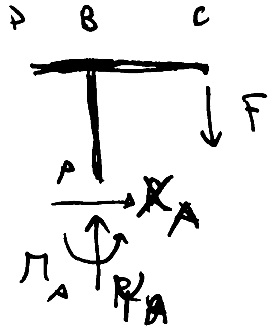
Crane



Step 1: Find a suitable design sketch and draw the free body diagram



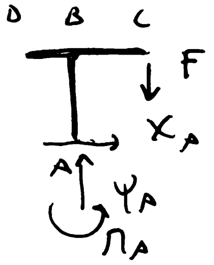
design sketch (simplification of the structure) ⇒ Free body diagram (restraints replaced by their reactions)



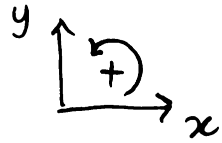
Step 2 Solve the problem of equilibrium, using the "cardinal" equations that state that a system of forces (like that given by the external forces — actions — and the reaction forces) is equilibrated if its resultant force is null and its resultant moment, about an arbitrary point, is null.

$$\begin{cases} \sum F_i = 0 \\ \sum M_{0,i} = 0 \end{cases} \longrightarrow \begin{cases} \sum X_i = 0 \\ \sum Y_i = 0 \\ \sum M_{0,i} = 0 \end{cases}$$

For a planar structure you have two equations for translations along two directions, and one equation for rotations about a pole



Reference system:



Positive moments: counterclockwise

$$\begin{cases} X_A + \text{nothing else} = 0 \\ Y_A + (-F) = 0 \\ M_A + (-F \cdot l_{BC}) = 0 \end{cases}$$

equilibrium with respect to translations about two axes

equilibrium with respect to rotations about the point A

We get:

$$\begin{cases} X_A = 0 \\ Y_A = F = 100 \text{ kN} \\ M_A = F \cdot l_{BC} = (100 \text{ kN}) \cdot (20 \text{ m}) = 2000 \text{ kN} \cdot \text{m} \end{cases}$$

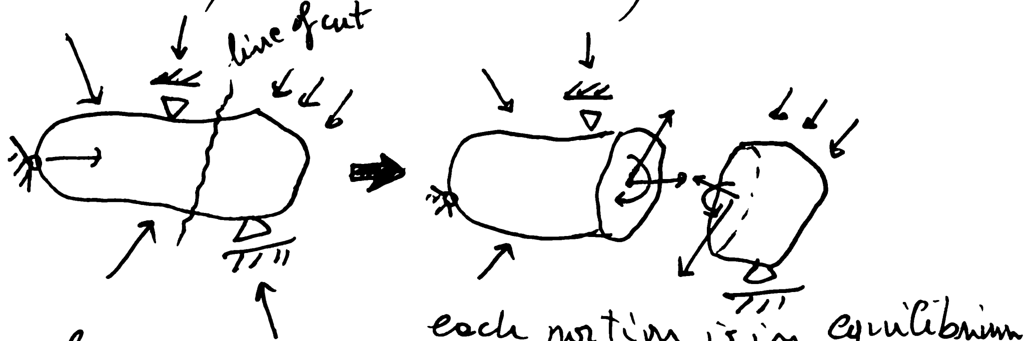
So reactions applied by the restraint in A are now a known element

step 3 find the internal forces

NOTE there is a principle that states that, if a generic body is in equilibrium under a certain set of ~~load~~ external forces and reactions, every single portion of the body, obtained by means of a cut that separates it from the remaining portion, is in equilibrium ~~and~~ under the external forces and reactions applied to that portion and under a system of forces that are mutually transmitted by the two faces of the cut.

In other words:

system in equilibrium

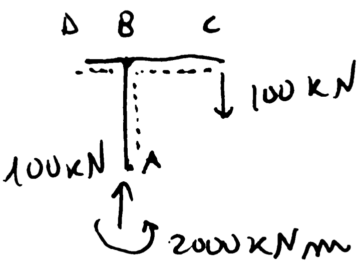


transmission of internal forces between the faces of a cut (i.e. a section)

each portion is in equilibrium if you consider a set of forces (internal forces) transmitted by the faces of the cut

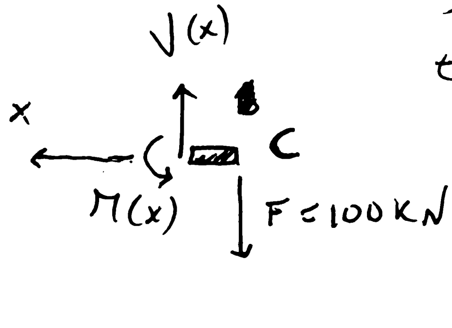
→ we do not care about their distribution, but only their resultant force and moment are taken into account

back to the example

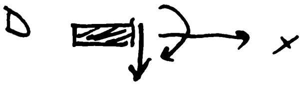


Internal forces: { Axial force N
Shear force V
bending moment M

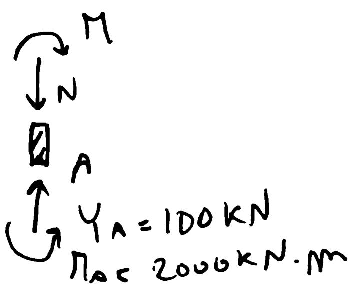
They are the resultants referred to a "standard" system with origin ~~in correspondence~~ in correspondance to the barycentre of the cross section, one direction tangent to the longitudinal axis of the member, and the second axis perpendicular to the first.



From C to B: $N = 0$
$V = F = 100\,kN > 0$
$M = -F \cdot x \leq 0$



From D to B: $V = 0$
$M = 0$
$N = 0$



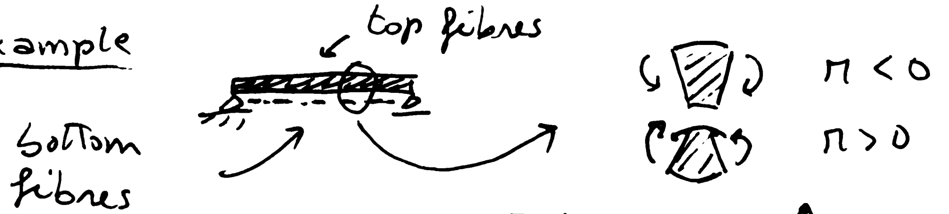
From A to B: $N = Y_A$ $(= F)$

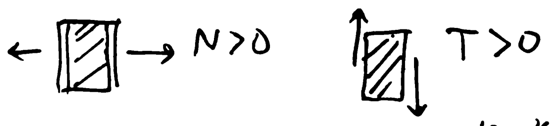
$M = M_A$

$V = 0$

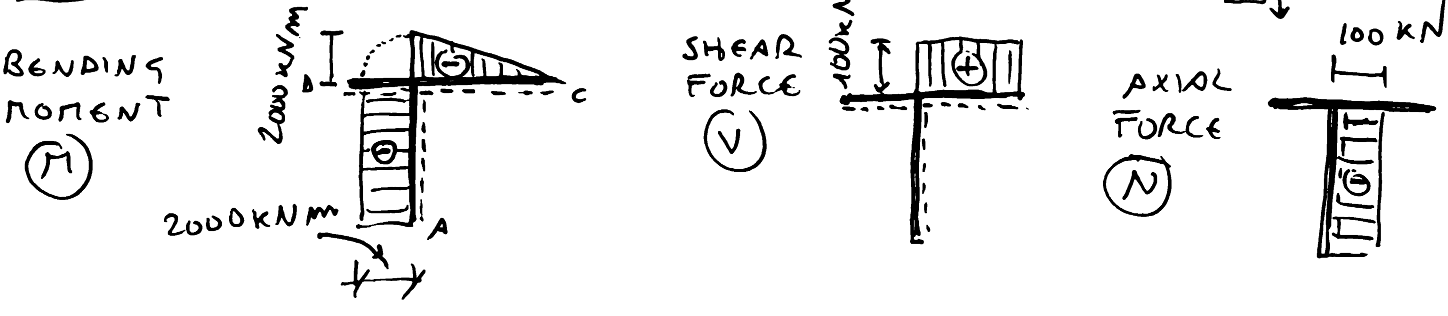
Now we can plot the diagrams of the internal forces, referred to our design sketch that reproduces the axis lines of the structure

Conventional rules : first, you have to decide wich side of your structure members should be considered as the bottom side (traced with a dashed line).
Then, plot the bending moment at the side "in tension", while shear and axial forces have to be plotted at the top side if positive, and at the bottom side if negative.
Moreover, N is positive ~~if~~ in case of tensile force, while V is positive when tends to rotate clockwise a portion of the beam. Example





back to the point



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● CLASSIFICATION OF STRUCTURES

In the previous example, a step was omitted: the classification of the structure with respect to its internal and external restraint conditions.
Indeed, it should be always preliminarly identified in which class the structure ~~can be~~ falls.

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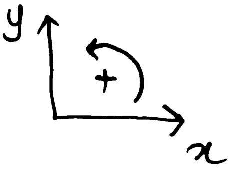
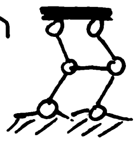
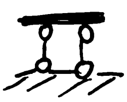
In general, you have that the structure is a system of t rigid bodies, connected by internal restraints and external restraints.
Each restraint, in a two-dimensional (plane) structure, can exert a force along a specific direction (it blocks one degree of freedom), a force along an arbitrary direction, wich can be divided into two components (it blocks two degrees of freedom), a bending moment (it blocks one degree of freedom – rotation instead of translation), or a combination of the previous. (NOTE: Degree of freedom → DOF)

COMMON RESTRAINTS

reference system

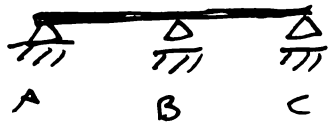


RESTRAINT		IT BLOCKS:	IT ALLOWS:	REACTIONS
ROLLER (simple → 1 D.O.F.)		translations along its axis	rotations, translations normal to its axis	F_y
PENDULUM (simple → 1 DOF)		like the roller		F_y
DOUBLE BI-PENDULUM (simple → 1 DOF)		rotations	translations along and perpendicularly to its axis	M
HINGE / PIN (double → 2 DOF)		translations (all)	rotation	$F=(F_x;F_y)$
BIPENDULUM (double → 2 DOF)		rotation; translation along its axis	translation perpendicular to the axis	F_y; M
FIXED END		translations (all); rotation	nothing	$F=(F_x;F_y)$ M

Note that each rigid body, in a two-dimensional problem, has three degrees of freedom (DOF): two related to translations, one related to rotation

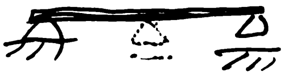
It is to be noted that not all the restraints could be really effective: some of them could be redundant, i.e. not strictly necessary for the equilibrium of a structure.
Morover, it cannot be taken for granted that a structure is stable if the number of restraints equates the number of degrees of freedom.

EXAMPLE



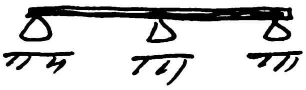
1 beam → 3 DOF
1 hinge (it blocks 2 DOF) + 2 rollers (1+1 DOF blocked) → 4 restraints

→ one of the two supports B, C is redundant



→ the equilibrium can be found the same

EXAMPLE



1 beam → 3 DOF
3 rollers → 3 restraints

→ but nothing blocks possible horizontal translations, so the structure is not stable

In general, we can say that a structure, made by t different rigid bodies mutually connected one to the other by internal restraints, and connected to the exterior by external restraints, has globally $3 \cdot t$ degrees of freedom.
The number of overall restraints is s (internal + external, and counting twice or three times double or triple restraints, respectively), of which s_{ef} is the number of effective restraints and i is the number of the ineffective (redundant) restraints. The number of possible movements of the structure (i.e. movements not blocked by the existing restraints, if any) is ℓ, called sometimes "degrees of lability".
We have:

$$\text{DoF} = 3 \cdot t$$

$$s = s_{ef} + i$$

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$$\ell = DOF - S_{ef} \overset{!}{=} 3 \cdot t - S_{ef} \overset{!}{=} 3t - S + i$$

(you have to count only the effective restraints)

$S = S_{ef} + i \rightarrow S_{ef} = S - i$

$$\Rightarrow \boxed{\ell - i = 3t - s}$$

the same number of restraints can be related to different situations

1) $\ell > 0$ and $i = 0$ [ipostatic] $\Rightarrow$ the structure ~~unstable~~ is unstable but statically determinate. Equilibrium exists for certain actions; in that case, the problem of finding reactions (and internal forces) can be solved

2) $\ell > 0$ and $i > 0$ $\Rightarrow$ the structure is unstable and statically indeterminate (redundant) Equilibrium exists for certain actions, but reactions cannot be calculated using equilibrium conditions.

3) $\ell = 0$ and $i = 0$ [isostatic] $\Rightarrow$ the structure is stable and statically determinate. Equilibrium always exists, and reactions can be found on the basis of equilibrium conditions.

4) $\ell = 0$ and $i > 0$ (hyperstatic) $\Rightarrow$ the structure is stable but statically indeterminate (redundant) Equilibrium exists but reactions cannot be found using only equilibrium conditions

Summary: only if there are no ineffective restraints (i.e. all the restraints are necessary and not redundant), that is $i = 0$, reactions can be found (of course, if equilibrium exists: always if $\ell = 0$, in certain situations if $\ell > 0$) using only equilibrium conditions.

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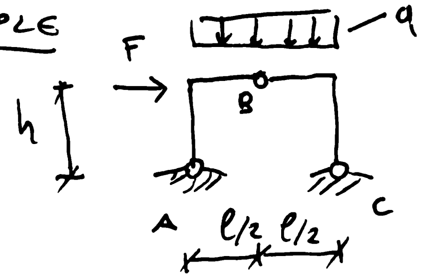
EXAMPLE



- three-hinge portal

Number of rigid bodies $t=2$ $\left(\Gamma + \rceil \right)$

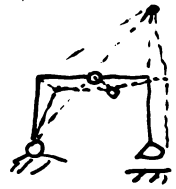
Overall degrees of freedom $DOF = 3t = 6$

Overall restraints s: two external hinges + an internal hinge → 6

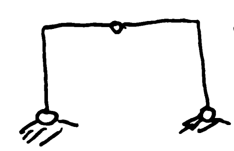
$s = 6$

Are there any ineffective restraint?

2 DOF not blocked →



1 DOF not blocked →



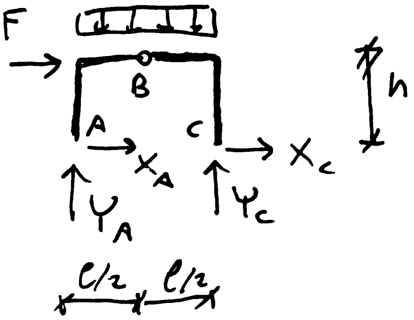
stable!

All the restraints are effective, the structure is stable $\Rightarrow l=0; i=0$

Therefore the structure is statically determinate

Free body diagram

ref. system



$$\begin{cases} (1)\ X_A + X_C + F = 0 & \text{Eq. horizontal translation} \\ (2)\ Y_A + Y_C - q \cdot l = 0 & \text{Eq. vertical translation} \\ (3)\ Y_C \cdot l - F \cdot h - (ql) \cdot \frac{l}{2} = 0 & \text{Eq. rotation about A} \\ (4)\ Y_C \cdot \frac{l}{2} + X_C h - \left(q\frac{l}{2}\right) \cdot \frac{l}{4} = 0 & \text{Eq. rotation of BC about B} \end{cases}$$

4 linearly independent equations, 4 unknown

→ the problem of equilibrium can be solved

(3) gives: $Y_C = F \cdot \frac{h}{l} + q\frac{l}{2} \longrightarrow Y_C$

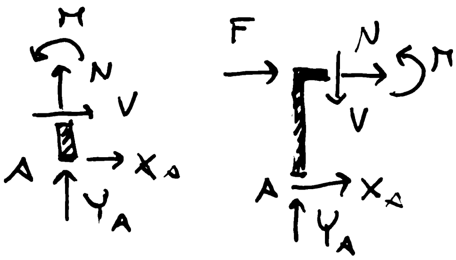
(2) gives: $Y_A + \left(F\frac{h}{l} + q\frac{l}{2}\right) = ql \longrightarrow Y_A = q\frac{l}{2} - F\frac{h}{l}$

(4) gives: $X_C = q\frac{l^2}{8} \cdot \frac{1}{h} - Y_C \frac{l}{2} \cdot \frac{1}{h} \longrightarrow X_C = -\left(\frac{ql^2}{4h} + \frac{F}{2}\right)$

(1) gives: $X_A = \frac{ql^2}{4h} - \frac{F}{2}$

Note that X_A can be 0 for $F = \frac{ql^2}{4h}$, or <0 for $F > \frac{ql^2}{4h}$

Once the reactions are calculated, internal forces (M, N, V) can be found in every point of the structure, by considering the equilibrium of proper portions.



etc.

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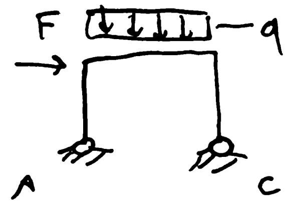
EXAMPLE



two-hinge portal

Number of rigid bodies: $t=1$

overall degrees of freedom $DOF=3$

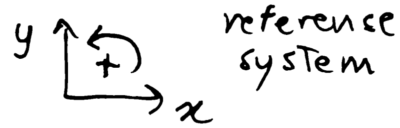
overall restraints: $2 \times 2 = 4 = S$

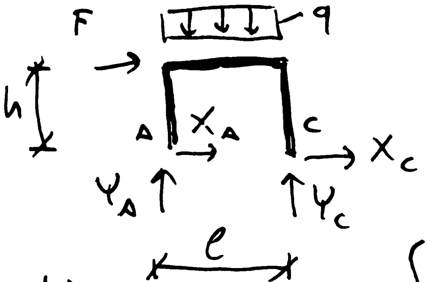
One of the restraints is ineffective

Which one? Depends upon your choice of a subset of restraints that give a stable and statically determinate structure

if you try to solve this structure (with reference to the problem of equilibrium and the calculation of internal forces), you will find that there are 3 equilibrium conditions that are linearly independent, but ~~are~~ 4 unknown.

Free body diagram





$$\begin{cases} X_A + X_C + F = 0 \\ Y_A + Y_C - q\ell = 0 \\ Y_C \cdot \ell - (q\ell) \cdot \frac{\ell}{2} - F \cdot h = 0 \end{cases}$$

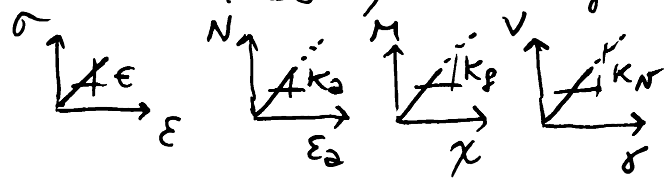
No way to find X_A and X_C separately, since we can get a linear system made by only 3 linearly independent equations, but we have 4 unknown quantities (4 reactions) $\rightarrow \infty^1$ solutions

$$\begin{cases} Y_C = F\,h/\ell + q\,\ell/2 \\ Y_A = q\,\ell/2 - F\,h/\ell \\ X_A + X_C = -F \end{cases}$$

The structure is statically indeterminate.

To deal with this kind of problems, we will need to find out how deformations can be related to the applied forces and to the internal forces, then to study the problem from both points of view (equilibrium and deformations).

The basic approach consists in the adoption of a linear elastic relation between tension and strain, and therefore between forces and deformations.



Then, it is possible to calculate deformations, thus obtaining more equations in order to cover the number of unknown by applying compatibility (congruence) conditions

FORCE METHOD $\rightarrow$ among the ∞^i equilibrated situations, find the only one that is compatible with the restraints

<u>DISPLACEMENT METHOD</u> among the ∞^{i} possible compatible situations, find the only one that is equilibrated with respect to the acting and reacting forces.

Any structural analysis that gives you the values of reactions and internal forces on the basis of linear elastic relations, is a linear analysis (1^{st} order analysis)

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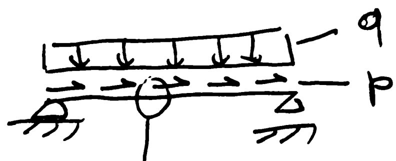
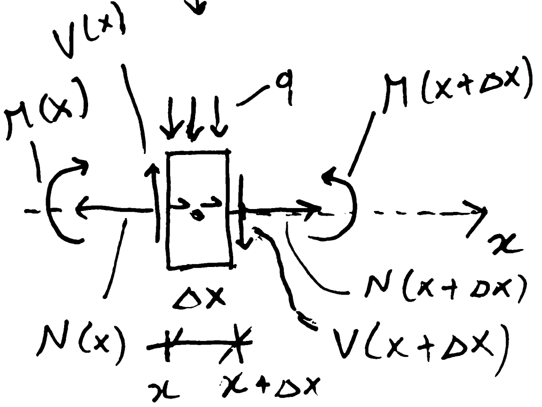
DIFFERENTIAL EQUATIONS FOR THE INTERNAL EQUILIBRIUM OF BEAMS (with <u>straight axis</u>)



looking at an infimitesimal portion of the beam



Assumptions: only uniformly distributed loads (ex. in kN/m, etc.)
straight axis

q = u.d.l. perpendicular to the axis of the beam

p = u.d.l. parallel to the axis of the beam

N = axial force referred to the standard reference system (barycentre of the cross section, principal axes of inertia of the cross section); tensile forces are positive

V = shear force referred to the st. ref. syst.

M = bending moment referred to the s.r.s.

1) Horizontal equilibrium:

$$\underbrace{N(x+\Delta x) - N(x)}_{\text{this is the variation } \Delta N} + \bar{p}\cdot\Delta x = 0$$

$\bar{p}$ → average value in Δx

$$\Delta N = -\bar{p}\,\Delta x \longrightarrow \frac{\Delta N}{\Delta x} = -\bar{p}$$

Considering the interval Δx that tends to 0 means that we can use limits, and therefore values calculated in discrete positions tends to a local value

$$\lim_{\Delta x \to 0} \frac{\Delta N}{\Delta x} = \frac{dN(x)}{dx} \qquad \lim_{\Delta x \to 0} \bar{p} = p(x)$$

$$\Longrightarrow \quad \frac{dN(x)}{dx} = -p(x)$$

2) Vertical equilibrium

$$\underbrace{V(x) - V(x+\Delta x)}_{\text{This is the opposite of the variation of } V} - \bar{q}\cdot\Delta x = 0$$

$\bar{q}$ is an average value in Δx

$$-\Delta V - \bar{q}\,\Delta x = 0 \longrightarrow \frac{\Delta V}{\Delta x} = -\bar{q}$$

Again, if $\Delta x \to 0$ we have $\lim\limits_{\Delta x\to 0} \dfrac{\Delta V}{\Delta x} = \dfrac{dV(x)}{dx}$; $\lim\limits_{\Delta x\to 0} \bar{q} = q(x)$

$$\Rightarrow \frac{dV(x)}{dx} = -q(x)$$

3) Equilibrium with respect to the rotation about the barycentre of the element

$$M(x+\Delta x) - M(x) - V(x+\Delta x)\cdot\frac{\Delta x}{2} - V(x)\frac{\Delta x}{2} = 0$$

Note that $N(x)$; $N(x+\Delta x)$; $q(x)$ and $p(x)$ do not give contributions since their resultants cross the barycentre of the element.

$$V(x+\Delta x) = V(x) + \Delta T \Rightarrow \Delta M - V(x)\frac{\Delta x}{2} - \Delta V\frac{\Delta x}{2} - V(x)\frac{\Delta x}{2} = 0$$

$$\Delta M - V(x)\,\Delta x - \Delta V\frac{\Delta x}{2} = 0$$

$\Delta V\cdot\Delta x$ can be disregarded compared to ~~di~~ the other quantities, since it is the ~~no~~ product of two infinitesimal components

$$\frac{\Delta M}{\Delta x} = V(x)$$

$$\Delta x \to 0 \Rightarrow \frac{dM(x)}{dx} = V(x)$$

deriving this equation we get: $\dfrac{d^2M(x)}{dx^2} = \dfrac{dV(x)}{dx} = -q(x)$

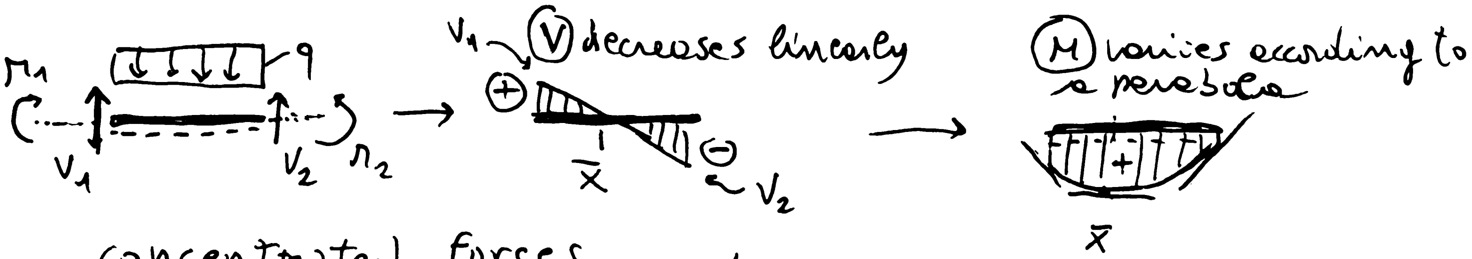
Summary:

$$\begin{cases} \dfrac{dN(x)}{dx} = -p(x) \\[2ex] \dfrac{dV(x)}{dx} = -q(x) \\[2ex] \dfrac{dM(x)}{dx} = V(x) \Rightarrow \dfrac{d^2M(x)}{dx^2} = -q(x) \end{cases}$$

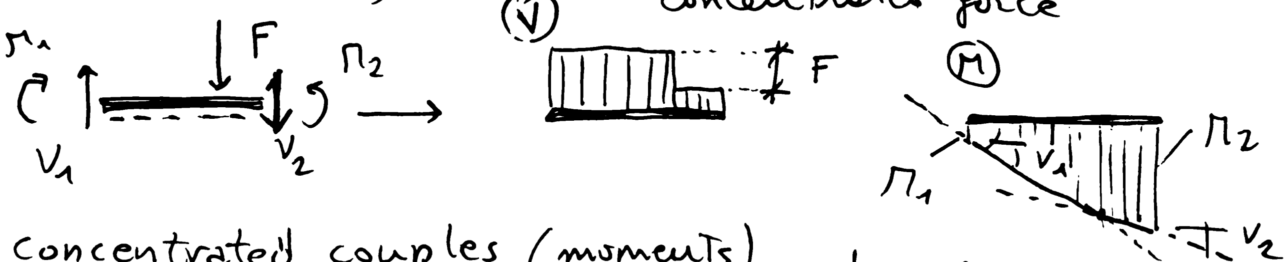
Thus: $V(x)$ constant means $q(x) = 0$ and implies a linear variation of $M(x)$

$p(x) = 0$ implies a constant axial force

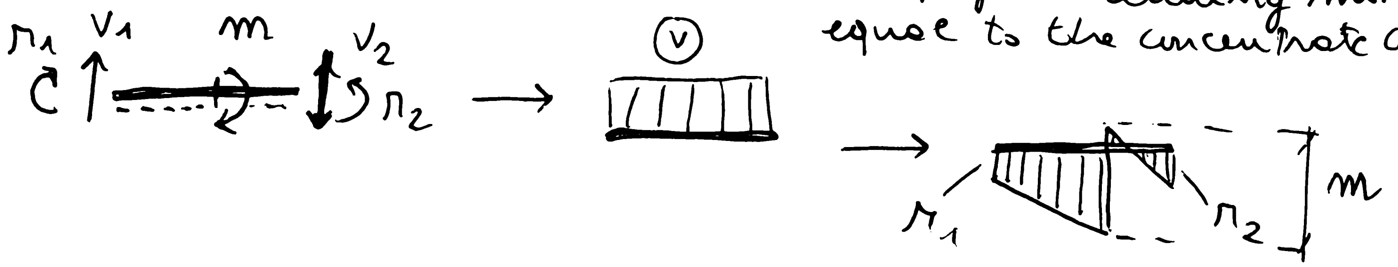
A constant $q(x)$ implies a linear variation of the shear force $V(x)$ and a parabolic variation of $M(x)$



concentrated forces (point loads) → drop of shear force equal to the concentrated force



concentrated couples (moments) → drop of the bending moment equal to the concentrate couple



TRUSSES (TRUSSED BEAMS)

Trusses are particular structures made by a composition of elementary modules (generally triangular) connected by hinges. Internally, they are statically determinate, and if the external restraints are sufficient and not redundant, they are globally stable and statically determinate.

By assumption, forces are applied only at the nodes, therefore each beam of the truss can be subjected only to normal compression forces (i.e. they are struts) or tensile forces (i.e. they are ties). Not bending moments nor shear forces affect the ~~beam~~ elements of a truss.

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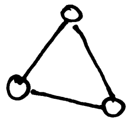
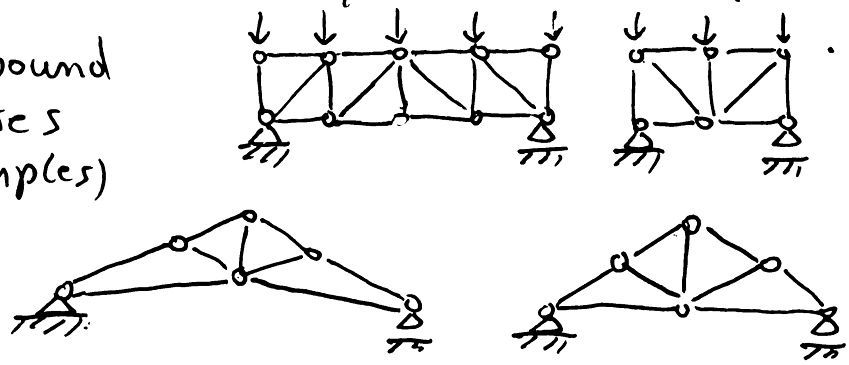
Indeed, internal hinges cannot withstand any moment, therefore (being forces applied only to the nodes, i.e. to the hinged connections) the variation of M is linear, and M is null at each end $\Rightarrow$ M is null all along the element!

simple truss



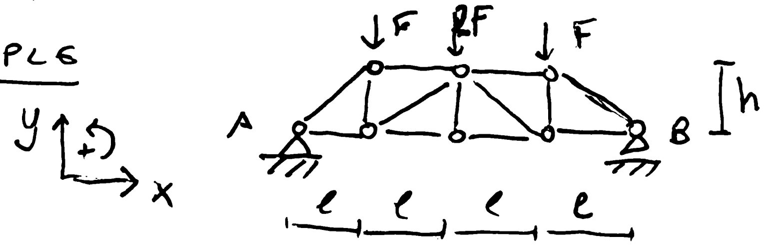
compound trusses (examples)



RESOLUTION typically, a truss is internally stable and statically determinate. If the external restraints are sufficient and not redundant, reactions can be found as usual.

To find the axial forces in the internal elements, there are several methods, like the equilibrium of the nodes and the Ritter's methods.

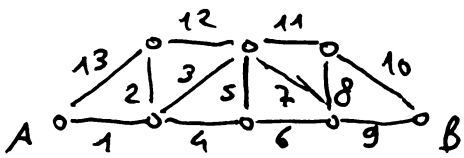
EXAMPLE



$X_A = 0$

$Y_A = Y_B = \frac{1}{2}(F+2F+F) \overset{!}{=} 2F$

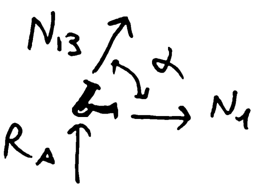
Truss members are usually numbered:



Equilibrium of the nodes

Node by node, internal forces are calculated on the basis of the equilibrium of a certain node

Node between 1 and 13 (point A)



vertical direction: $R_A + N_{13} \cdot \sin\alpha = 0$

$N_{13} = -R_A/\sin\alpha = -2F/\sin\alpha$

We assumed N_{13} as a tensile force, but the equilibrium gives a minus, therefore our assumption was wrong $\rightarrow$ N_{13} is a compression force!

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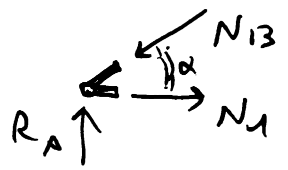
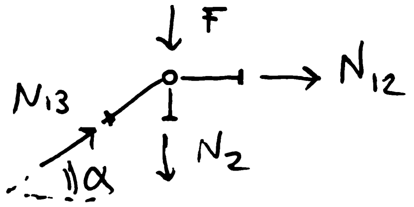
horizontal direction (known that N_{13} is a compression force)



$N_1 - N_{13} \cdot \cos\alpha = 0 \rightarrow N_1 = N_{13} \cos\alpha \overset{!}{=} 2F \cot\alpha$

NOTE: positive sign of N_1 means only that our assumption about the direction was right

second node: between 13 and 12



N_{13} → compression; already calculated

Eq. horizontal direction

$N_{12} + N_{13} \cos\alpha = 0 \rightarrow N_{12} = -N_{13} \cos\alpha \overset{!}{=} -2F \cot\alpha$

We chose the wrong direction, N_{12} is the opposite of our assumption

Vertical direction

$F + N_2 = N_{13} \sin\alpha \rightarrow N_2 = N_{13} \sin\alpha - F$

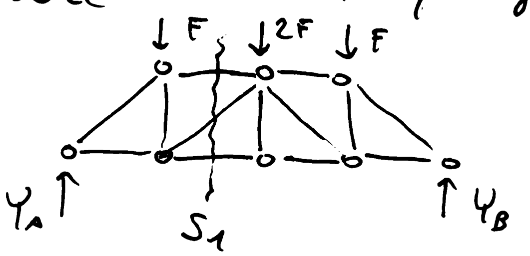
$N_2 = \frac{2F}{\cancel{\sin\alpha}} \cdot \cancel{\sin\alpha} - F = F$ (ok, right assumption)

And so on. It is thus possible to find each internal force

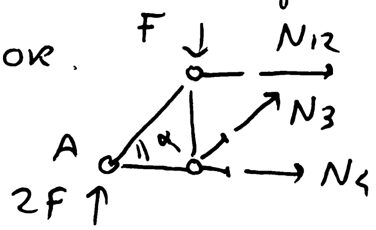
Ritter's method

This method consists in making proper cuts, thus sectioning the beam in substructures.

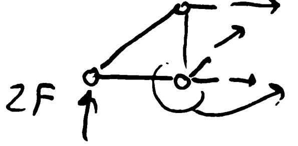
A proper ~~cut~~ section must cut (at least) three beams that are not crossing together at the same point



S_1 is ok.



Then, we have just to consider the ~~...~~ rotational equilibrium about clever points (i.e., points where ~~one~~ two beams intersect, and not the third one)



equilibrium about this point to find N_{12}

$N_{12} \cdot h + 2F \cdot \ell = 0$

$N_{12} = -2F \cdot \ell / h = \underline{-}2F \cot\alpha$

wrong direction

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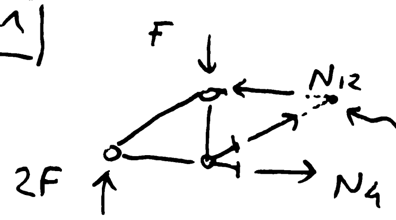


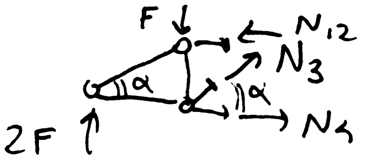
equilibrium about the intersection of beam 12 and beam 3 to find N_4

$N_4 \cdot h + F \cdot \ell - 2F \cdot 2\ell = 0$

$N_4 = 3F\,\ell/h \leftarrow \ell/h = \cot\alpha$

$\overset{!}{=} 3F\cot\alpha$

Eq. in vertical direction



$N_3 \cdot \sin\alpha + 2F - F = 0 \rightarrow N_3 = -F/\sin\alpha$

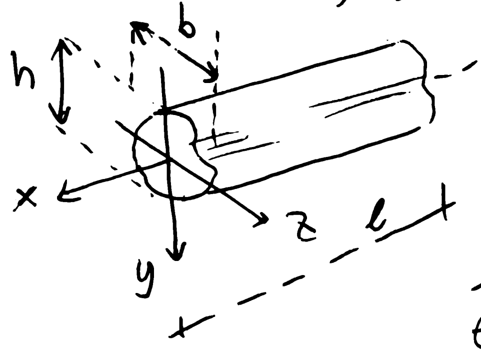
compression

And so on!

THE EULER-BERNOULLI BEAM

This is the most popular model that applies to monodimensional beams with a prismatic shape, and is based on a linear elastic relation between forces and deformations.

A beam is defined as a solid and homogeneous body which has two sizes of the same order of magnitude and one size sensibly greater (an order of magnitude bigger).



b, h comparable

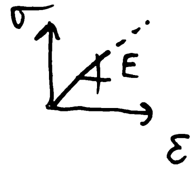
$\ell \gg b; h$ (about 5-10 times and more)

A beam is then defined by the axis line, i.e. the line described by the barycenter of the subsequent cross sections.

By assumption, the shear deformation is negligible and, therefore, not taken into account by this model

The model consider a linear elastic relation between the local material strain ε and the local normal stress σ, and ~~demonstrates~~ assumes that the bending moment M is proportional to the curvature χ of the beam, and that the axial force N is proportional to the axial deformation ε_a

locally :



slope of the line: E (Young's modulus)

$\Rightarrow$ $N = K_a \cdot \varepsilon_a$ — K_a → constant; ε_a → axial deformation

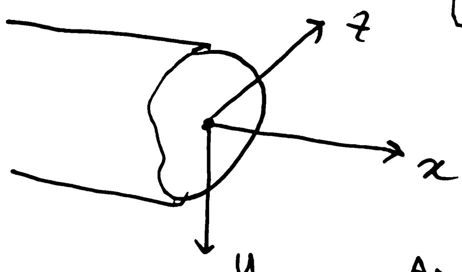
$M = K_f \cdot \chi$ — K_f → constant; χ → curvature

K_a is called axial stiffness of the beam

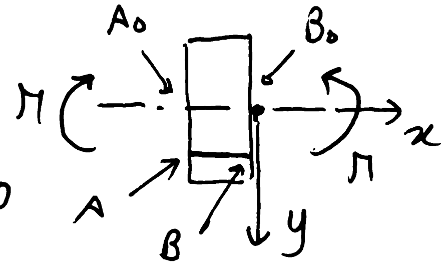
K_f is called bending stiffness of the beam

1 – Let's consider no axial force, first: therefore, $\varepsilon_a = 0$ (no axial deformation)

assumption → $\boxed{\begin{matrix} N = 0; \ \varepsilon_a = 0 \\ M \neq 0; \ \chi \neq 0 \end{matrix}}$ along the length of the beam



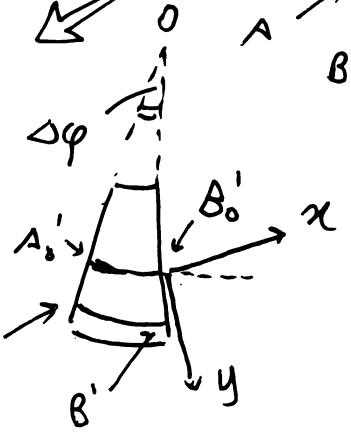
Let consider a portion of beam located at a certain position x, subjected to a bending moment M



A_0, B_0 → axial segment

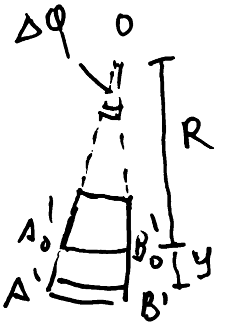
A, B → segment located at a distance y from the axis

Deformed situation



Being small the deformation, $A_0'B_0'$ and $A'B'$ can be approximated by a straight line

By assumption, $\overline{A_0'B_0'} = \overline{A_0B_0}$ (no longitudinal deformation along the axis)



$$\Delta\varphi = \frac{\overline{A_0'B_0'}}{R} = \frac{\overline{A'B'}}{R+y} \quad \text{and} \quad \overline{A_0'B_0'} = \overline{A_0B_0}$$

The longitudinal strain at a position y, $\varepsilon(y)$, is defined as: $\varepsilon(y) = \dfrac{\overline{A'B'} - \overline{AB}}{\overline{AB}}$

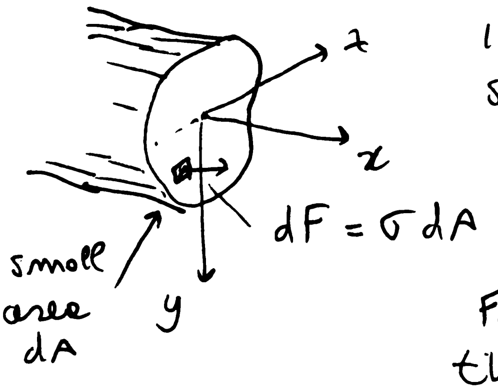
Initially, all the fibres of the beam have the same dimension since there is no deformation: $\overline{AB} = \overline{A_0B_0}$

Thus $$\varepsilon(y) = \frac{(R+y)\Delta\varphi - R\Delta\varphi}{R\Delta\varphi} = \frac{R+y-R}{R} = \frac{y}{R} = y \cdot \chi$$

by definition, $\chi = \dfrac{1}{R}$

Being the curvature defined as the inverse of the radius of curvature, we get: $\varepsilon(y) = y \cdot \chi$

This is valid along a cross section placed at a position x



If we consider a small area on the cross section, there is a small force $dF = \sigma\, dA$ acting there, due to the presence of a normal stress (being $M \neq 0$ by assumption)

For simplicity, we consider a plane problem, thus there is only one possible bending moment M_z about z ($M_z \rightarrow M$ because it's the only one)

By definition: $M = \int_A dF \cdot y$ → distance from the axis z

the whole cross section → A

Pay attention to the sign: a moment that causes tension on the bottom side of the beam is positive for us, but is directed in the opposite direction with respect to our choice of the axis z → positive σ (tension) and positive y → positive contribution to M

$$M = \int_A \underbrace{\sigma \cdot dA}_{dF} \cdot y \longrightarrow \sigma = E \cdot \varepsilon$$

by assumption

local property: σ, ε change along the vertical direction

$$= \int_A E \cdot \varepsilon(y) \cdot y \cdot dA \longrightarrow \varepsilon(y) = y \cdot \chi$$

$$= \int_A E \cdot y \cdot \chi \cdot y\, dA \longrightarrow M = E \cdot \chi \cdot \underbrace{\int_A y^2 dA}$$

E → constant

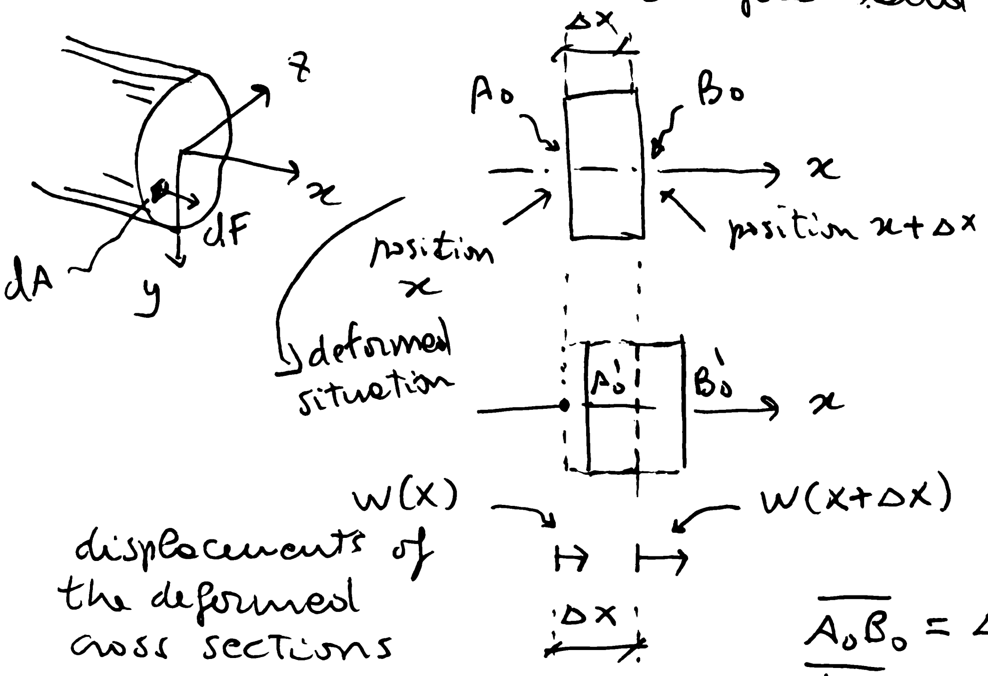
χ → constant in a certain cross section at x

definition of the second moment of area (moment of inertia)

$$\Rightarrow M(x) = \underbrace{E \cdot I}_{K_f} \cdot \chi(x)$$

K_f, ~~axial~~ bending stiffness

2) Let's consider now $M=0$, $\chi=0$, ~~therefore~~ and $N\neq 0$, $\varepsilon_a \neq 0$



Be w(x) the function that describes the displacement of the cross section due to the axial deformation

The axial strain ε_a is: $\varepsilon_a = \dfrac{\overline{A_0'B_0'} - \overline{A_0B_0}}{\overline{A_0B_0}}$

$$\overline{A_0B_0} = \Delta x$$
$$\overline{A_0'B_0'} = \Delta x + w(x+\Delta x) - w(x)$$

By definition, $N = \int_A dF = \int_A \sigma \, dA$ (the whole cross section → A)

$\sigma = E \cdot \varepsilon$ (elastic hypothesis)

$\varepsilon = \varepsilon_a$ for every fibre

$$\rightarrow \quad N = \int_A E\cdot\varepsilon \cdot dA = E\cdot\varepsilon_a\cdot\int_A dA$$

this is simply the area of the cross section

$$N(x) = \underbrace{E\cdot A}_{k_a,\ \text{axial stiffness}}\cdot\,\varepsilon_a(x)$$

Summary: EULER-BERNOULLI MODEL FOR THE BEAM IN BENDING

$$\left\{\begin{array}{l}\text{linear elastic material} \\ (\sigma/\varepsilon = E) \\ \text{no shear deformation} \\ \text{(negligible}\end{array}\right. \Rightarrow \begin{array}{l} N(x) = E\cdot A\cdot\varepsilon_a(x) \\ M(x) = E\cdot I\cdot\chi(x)\end{array}$$

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KINEMATICS OF THE BEAM

A beam is generally described by its axis line, which is the line made by the barycentre of the cross sections, and which is perpendicular to each cross section.

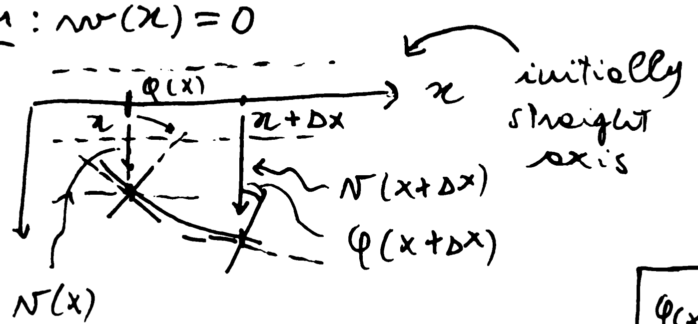
The deformed shape of a beam can be described (in a plane problem) by three functions:

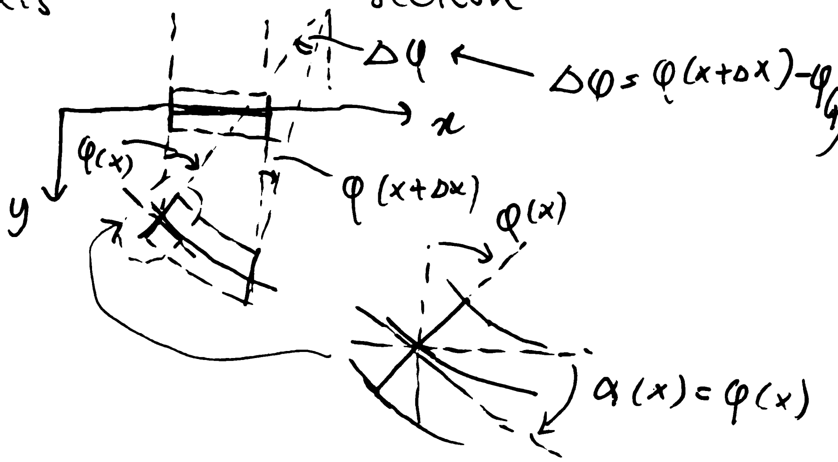
$w(x) \rightarrow$ horizontal displacement

$v(x) \rightarrow$ vertical displacement

$\varphi(x) \rightarrow$ rotation of the cross section

CASE 1 : $w(x) = 0$



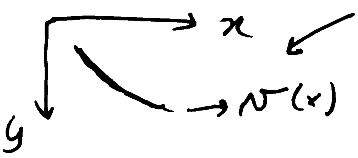


In the first case, we consider only vertical displacements and rotations of the cross section, without horizontal displacements $w(x)$

By assumption : $\varphi(x) > 0$ if counterclockwise $+\circlearrowleft$

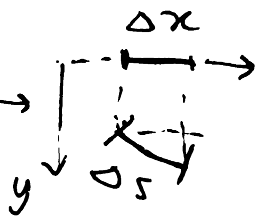
$$\alpha(x) = \varphi(x) \cong -\frac{dv(x)}{dx}$$

positive $v'(x)$ means clockwise rotations (negative for us)



Such a curve has first derivative positive, second derivative negative

Δs = deformed length Δx $\longrightarrow$



$$\Delta s \cong \sqrt{\Delta x^2 + \Delta y^2}$$

$$\Delta s \cong \sqrt{\Delta x^2 + \left(\frac{\Delta y}{\Delta x} \cdot \Delta x\right)^2}$$

$$\Delta s = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$\Delta \varphi \cong \Delta s / R$ ← average value in Δs for the radius of curvature

$$\frac{\Delta \varphi}{\Delta s} \cong \frac{1}{R} \qquad \frac{\Delta \varphi}{\Delta x} \cdot \frac{\Delta x}{\Delta s} \cong \frac{1}{R}$$

Using limits, $\Delta x \rightarrow 0 \Rightarrow \dfrac{d\varphi(x)}{dx} \cdot \dfrac{dx}{ds} = \dfrac{1}{R(x)}$

$$ds = dx\sqrt{1 + [v'(x)]^2}$$

$\dfrac{1}{R(x)} = \chi(x)$ definition of curvature

$$\chi(x) = \frac{d\varphi(x)}{dx} \cdot \frac{1}{\sqrt{1+[v'(x)]^2}}$$

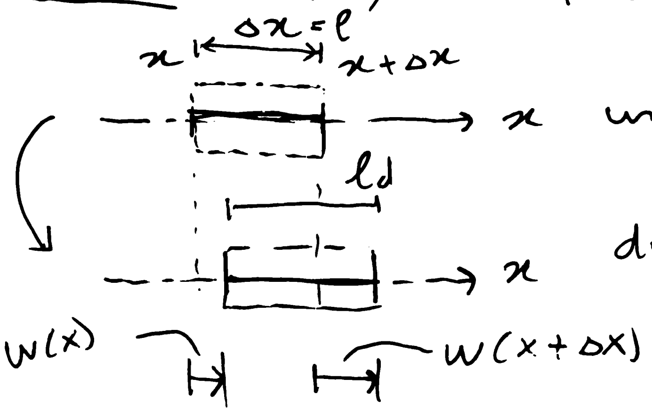
Actually, if by assumption deformations are small, $v'(x)$ (that is, $\frac{dv(x)}{dx}$) is small and its square is an infinitesimal that can be neglected. $\longrightarrow$ $\chi(x) = \frac{d\varphi(x)}{dx} = -\frac{d^2v(x)}{dx^2}$

Pay attention to the sign.
Tipically, we want that the curvature has a positive value if the second derivative is negative:

we want χ positive this way

$\left(\frac{d^2v}{dx^2} < 0\right)$

CASE 2: $w \neq 0$; $v(x) = \varphi(x) = 0$



axial deformation ε_a:

$$\varepsilon_a = \frac{l_d - l}{l}$$

l_d = deformed length

l = initial length

$l = \Delta x$!

$$\varepsilon_a = \frac{\overbrace{[\Delta x + w(x+\Delta x) - w(x)]}^{l_d} - \overbrace{\Delta x}^{l}}{\Delta x} = \frac{\Delta w}{\Delta x}$$

Using limits, $\Delta x \to 0$ $\Rightarrow$ $\varepsilon_a(x) = \frac{dw(x)}{dx}$
and you get a local value

<u>SUMMARY</u> Under the hypotheses of the Euler-Bernoulli beam, you can obtain the following relations between curvature $\chi(x)$, axial strain $\varepsilon_a(x)$, vertical displacements $v(x)$, horizontal displacements $w(x)$, cross sectional rotations $\varphi(x)$

$$\begin{cases} \varphi(x) = -\dfrac{dv(x)}{dx} \\ \chi(x) = \dfrac{d\varphi(x)}{dx} = -\dfrac{d^2v(x)}{dx^2} \\ \varepsilon_a(x) = \dfrac{dw(x)}{dx} \end{cases}$$

which can be combined with:

$$\begin{cases} M(x) = E \cdot I \cdot \chi(x) \\ N(x) = EA \cdot \varepsilon_a(x) \end{cases}$$

in order to calculate the deformed shape of a linear elastic structure and to solve a linear elastic statically indeterminate structure

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