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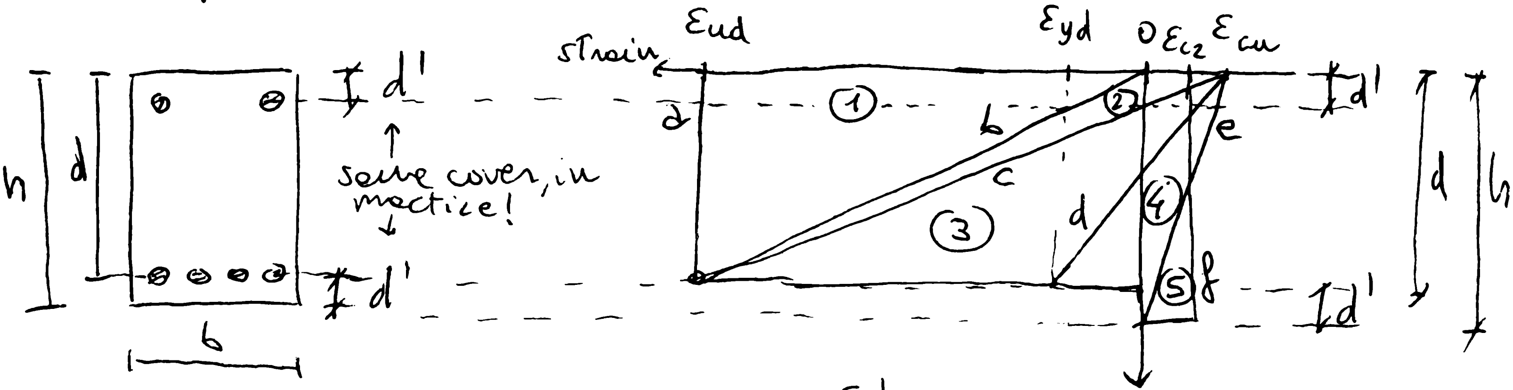
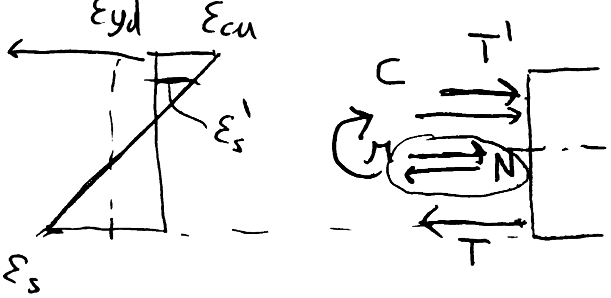
INTERACTION DIAGRAMS FOR COMBINED BENDING AND AXIAL FORCE

As seen in the previous lecture, all the possible combinations of bending moment M and axial force N that a rectangular section (for simplicity) with single or double reinforcement can withstand, are included between certain limit lines, which indicate five failure fields related to various situations.



EXAMPLE
Generic cross-sectional deformation corresponding to the ultimate capacity of a cross section in field (3), for example



Depending upon the amount of steel at the bottom and at the top part of the beam, the ~~ex~~ internal forces corresponding to this situation can be: simple bending ($N=0$), if $C+T'=T$; combined bending and tension ($N>0$), if $T>C+T'$; combined bending and compression ($N<0$), if $C+T'>T$.

Each field, from (1) to (5), corresponds to a specific range, which contains the possible cross sectional deformations that can be described by ~~the~~ position of the neutral axis.

Being $y = \xi \cdot d$ the neutral axis depth (ξ is simply defined as y/d), we could derive analytical expressions in order to calculate the combination of N and M corresponding to each position of the cross section.

In other words, if we make vary ξ (i.e. the depth of the neutral axis) ~~keep~~ and we keep constant a failure condition (i.e. the maximum strain ε_{ud} for steel, in fields (1) and (2), and

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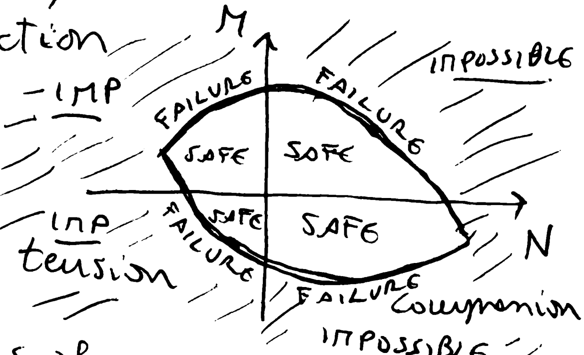
the maximum strain in concrete ε_{cu}, for fields ③ and ④, and the maximum average strain for the whole section in compression, ε_{c2}, for field ⑤), we would be able to get a continuous function that relates N and M, which should be considered as N_{Rd} and M_{Rd}, since we consider design properties for materials and we impose one or more failure conditions [Rd stands for resistance, capacity).
Therefore, if we consider a continuous variation of ξ from $-\infty$ to $+\infty$, thus from the limit line a to the limit line f, and we repeat the procedure for an opposite direction of bending moment M (i.e. reversing the concept of top and bottom steel), we can obtain what is called "interaction diagram" for bending moment and axial force.
The interaction diagram M-N is a closed and convex curve in the plane M, N which represents the failure conditions for a given cross section, and which delimits the safe domain of that cross section.



The limit curve represents failure, all the possible combinations of N and M that fall inside the domain are safe, while outside the domain there are combinations of M and N that cannot be withstood by the given cross section.

NOTE In practice, since those domains (i.e. the interaction diagrams) are used mainly for columns, where the axial force is a compression force, they consider (for simplicity) N positive (right side of the diagram) if compression (differently from the usual conventions which assume tension >0)

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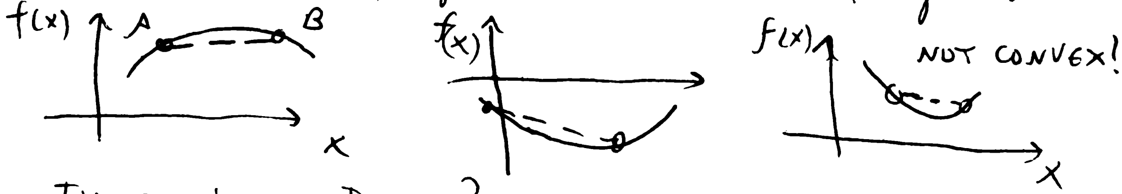
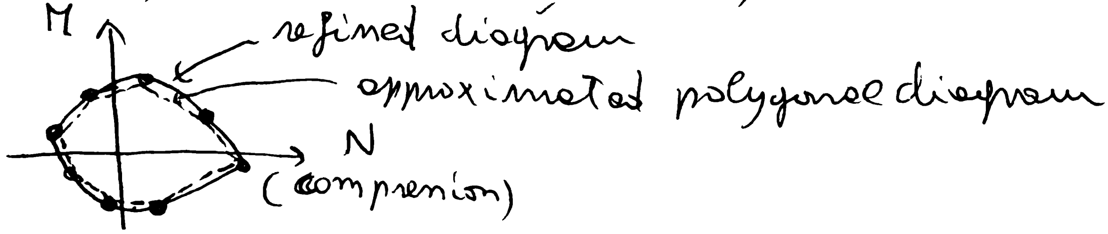
Beside being closed, the interaction diagram that can be calculated with refined ~~ana~~ analytical procedures is also convex.

If we consider a generic function, we can say that it is convex if it is possible to connect with a straight line two arbitrary points, and that line ~~is~~ stands ~~below~~ below the curve (or above, if we are on the negative semiplane)



Why is this property so important?
Because it allows us to calculate just few important points that belong to ~~the~~ the general domain N-M, instead of finding the more general functions related to a continuous variation of ξ, and then to connect those points with straight segments.
Therefore, we can get this way an approximated polygonal diagram which is, however, contained by the more precise one.



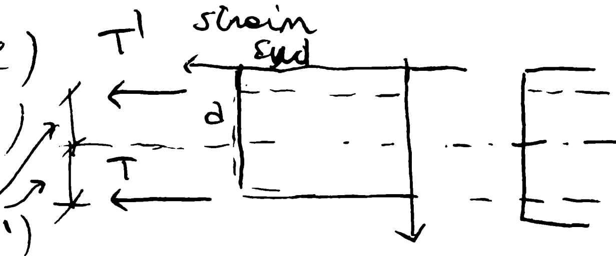
Of course, the significant points that we are looking for, correspond to the usual limit lines a, b, c, d, e and f already seen.

1) Point A (related to line a)

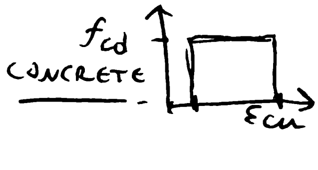
ε_s (strain of the ~~top~~ bottom steel)

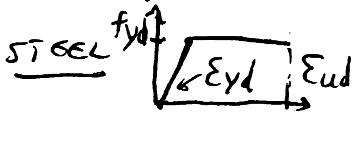
ε'_s (strain of the top steel)

$\rightarrow \varepsilon_s = \varepsilon'_s = \varepsilon_{ud}$



NOTE From now onward, we adopt, for simplicity, a stress-block law for concrete and a law without hardening for steel





in this case, we have all the section in tension ($\varepsilon > 0$ everywhere), so no contribution of concrete (as usual, we consider that concrete does not react in tension → conventional II state). Moreover, all the steel is yielded, therefore the force in steel is known, and equal to the product of the area and the yielding design strength.

$$\rightarrow N_{Rd,A} \overset{!}{=} -f_{yd} A_s - f_{yd} \cdot A'_s = -f_{yd}(A_s + A'_s)$$

A_s → area of bottom steel

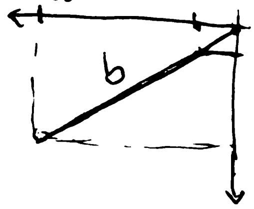
A'_s → area of top steel

$$\rightarrow M_{Rd,A} = (f_{yd} \cdot A_s - f_{yd} A'_s) \cdot \left(\frac{h}{2} - d'\right) \overset{!}{=} f_{yd}(A_s - A'_s) \cdot \left(\frac{h}{2} - d'\right)$$

$$\boxed{\begin{aligned} N_{Rd,A} &= -f_{yd} \cdot (A_s + A'_s) \\ M_{Rd,A} &= f_{yd}(A_s - A'_s)\left(\frac{h}{2} - d'\right) \end{aligned}}$$

NOTE: as said before, from a practical point of view it is preferable to work with axial forces that are positive in compression, instead that positive in tension as ~~they~~ should be. $M_{Rd,A} = 0$ if $A_s = A'_s$ (symmetric reinf.)

2) POINT B (related to the line b)

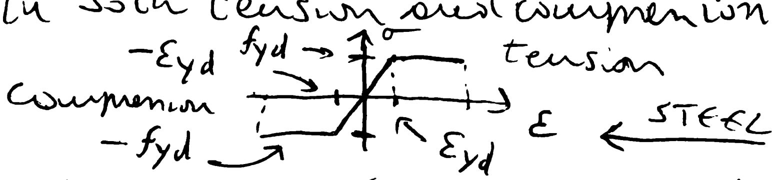


$\varepsilon_s = \varepsilon_{ud}$

we need to calculate ε'_s, that is positive (tension), but we need to check if ε'_s is greater than ε_{yd} (reinforcement yielded) or lower (elastic range)

$$\frac{\varepsilon'_s}{\varepsilon_{ud}} = \frac{d'}{d} \rightarrow \varepsilon'_s = \frac{d'}{d} \cdot \varepsilon_{ud}$$

In most practical cases, the top steel would be yielded in tension (do not forget that we assume, for steel, the same behaviour in both tension and compression!)



EXAMPLE: steel class C, $f_{yk} = 500$ MPa

$$\varepsilon_{yd} = \frac{f_{yk}/1.15}{E_s} \overset{!}{=} 2.17‰ = 0.00217$$

E_s → 200 GPa

$$\varepsilon_{ud} = 0.9 \cdot \varepsilon_{uk} = 0.9 \times 7.5\% \overset{!}{=} 6.75\% = 0.0675$$

Reasonably low value for: $\frac{d'}{d} = \delta = 5\%$ (0.05)

$$\rightarrow \varepsilon'_s = 0.05 \cdot 67.5‰ = 3.375‰$$

$$\varepsilon'_s > \varepsilon_{yd}$$

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However, being $\delta = \frac{d'}{d}$, we can obtain, from the equilibrium of the cross section:

$N_{Rd,B}$ = force in the bottom steel + force in the top steel (compression is positive!)

$$\stackrel{!}{=} \underbrace{-f_{yd}\cdot A_s}_{\substack{\text{bottom } \cancel{\text{top}} \text{ steel yielded}\\ \text{in tension}}} \underbrace{- f_{yd}\cdot A'_s}_{\substack{\text{top steel yielded}\\ \text{in most cases}}} = -f_{yd}\,(A_s + A'_s) = N_{Rd,A}$$

$\rightarrow$ check $\varepsilon'_s = \frac{d'}{d}\cdot \varepsilon_{ud}$!

the most general expression is: $\boxed{N_{Rd,B} = -f_{yd}\cdot A_s - \sigma'_s \cdot A'_s}$

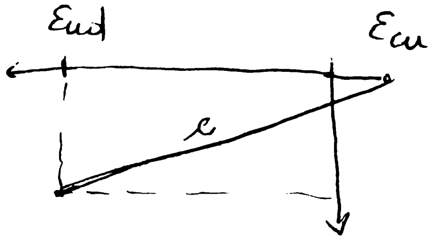
with $\sigma'_s = \begin{cases} f_{yd} & \text{if } \frac{d'}{d}\varepsilon_{ud} \geq \varepsilon_{yd} \\ E_s\cdot\varepsilon'_s = E_s\cdot\frac{d'}{d}\varepsilon_{ud} & \text{otherwise} \end{cases}$

$$\boxed{M_{Rd,B}} = \Big(\underbrace{+f_{yd}\cdot A_s}_{\substack{\text{positive contribution}\\ \text{of the bottom steel in}\\ \text{tension}}} \underbrace{- \sigma'_s\cdot A_s}_{\substack{\text{negative contribution of}\\ \text{the bottom steel in tension}}}\Big)\cdot\left(\frac{h}{2} - d'\right)$$

$\rightarrow$ they have the same lever arm

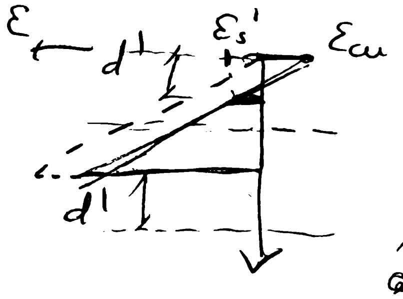
if $\delta\cdot\varepsilon_{ud} \geq \varepsilon_{yd}$, we get $M_{Rd,B} = f_{yd}\,(A_s - A'_s)\left(\frac{h}{2} - d'\right) \stackrel{!}{=} M_{Rd,A}$

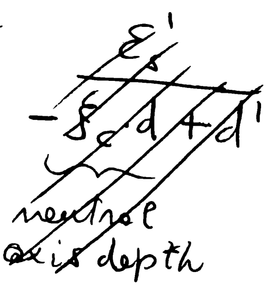
3) Point C
(line C)



the bottom steel is yielded. The strain of the top steel can be derived from similar triangles

To find ε'_s:
let assume that $\varepsilon'_s > 0$, and consider only absolute values





$$\frac{\varepsilon'_s}{d' - \underbrace{\xi_c d}_{\text{neutral axis depth}}} = \frac{\varepsilon_{ud} + \varepsilon_{cu}}{d}$$

of course, if this denominator is ~~would be~~ negative, therefore our assumption is not true and the top steel is compressed instead of under tension

$$\varepsilon_s' = \frac{d' - \xi_c d}{d}(\varepsilon_{cu} + \varepsilon_{ud}) = \Big(\underbrace{\frac{d'}{d}}_{\delta} - \xi_c\Big)(\varepsilon_{cu} + \varepsilon_{ud})$$

Alternatively: $\dfrac{\varepsilon_s'}{d' - \xi_c d} = \dfrac{\varepsilon_{cu}}{\xi_c \cdot d} \rightarrow \varepsilon_s' = \dfrac{\delta - \xi_c}{\xi_c} \cdot \varepsilon_{cu}$

EXAMPLE $\varepsilon_{cu} = 3.5‰$ $(\leq C50/60)$ $\qquad \xi_c = \dfrac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_{ud}}$

$f_{yk} = 500$ MPa, class C $\rightarrow \varepsilon_{ud} = 67.5‰$

$$\xi_c = \frac{3.5}{3.5 + 67.5} = 0.0493$$

$\delta = 0.05 = \dfrac{d'}{d} \rightarrow \varepsilon_s' \simeq 0$

$\rightarrow$ for common values of δ, the top steel is in tension, even yielded for greater values of d'.

case 1 $\rightarrow$ $\delta < \xi_c \rightarrow \varepsilon_s' < 0$ top steel in compression

case 2 $\rightarrow$ $\xi_c \leq \delta < \xi_c + \dfrac{\xi_c}{\varepsilon_{cu}} \cdot \varepsilon_{yd} \rightarrow$ top steel in tension, $\sigma_s' = E_s \cdot \varepsilon_s'$

case 3 $\rightarrow$ $\xi_c + \dfrac{\xi_c}{\varepsilon_{cu}} \cdot \varepsilon_{yd} \leq \delta \rightarrow$ top steel yielded in tension $\sigma_s' = f_{yd}$

$$\boxed{N_{Rd,c}} = \underbrace{0.8\, \underbrace{\xi_c d}_{y_c = \text{neutral axis depth}} \cdot b \cdot f_{cd}}_{\text{concrete compression force}} - f_{yd} \cdot A_s - \sigma_s' \cdot A_s'$$

because $\delta - \xi_c < 0$

1) if $\delta < \xi_c \rightarrow \varepsilon_s' < 0 \rightarrow \sigma_s' < 0$ then the contribution (compression) is positive

2) if $\xi_c \leq \delta < \xi_c + \dfrac{\xi_c}{\varepsilon_{cu}} \varepsilon_{yd} \rightarrow \sigma_s' = \varepsilon_s' E_s$

3) if $\delta \geq \xi_c + \dfrac{\xi_c}{\varepsilon_{cu}} \varepsilon_{yd} \rightarrow \sigma_s' = f_{yd}$ top steel yielded in tension

NOTE for tension σ we keep the usual conventions:
$\sigma > 0$ $\rightarrow$ tension!
$\varepsilon > 0$

$$\boxed{M_{Rd,c}} = 0.8 \xi_c d \cdot b f_{cd} \left(\frac{h}{2} - 0.4 \xi_c d\right) + \left(f_{yd} A_s - \sigma_s' A_s'\right)\left(\frac{h}{2} - d'\right)$$

if $\sigma_s' < 0 \rightarrow$ compression $\rightarrow$ positive contrib. to the moment

if $\sigma_s' > 0 \rightarrow$ negative contribution to the bending moment!

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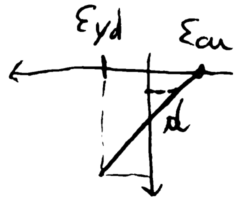
4) Point D (related to line d)



The bottom steel is yielded, so its tension is f_{yd}

ε_{yd} is usually lower, in absolute value, than ε_{cu}

To reach $\varepsilon_{yd} = 3.5‰$, you need a strength of $E_s \cdot (3.5‰) = 700$ MPa
(E_s = 200 GPa)

Thus, at least half of the section is in compression $\longrightarrow$ $\xi_d \geq 0.5$

We can conclude that the top steel is in compression

$$\frac{\varepsilon_s'}{\xi_d \cdot d - d'} = \frac{\varepsilon_{cu}}{\xi_d \cdot d} \rightarrow \varepsilon_s' = \frac{\xi_d - \delta}{\xi_d} \cdot \varepsilon_{cu} \qquad \xi_d = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_{yd}} \quad \left[\delta = \frac{d'}{d}\right]$$

To have the top steel in the elastic range (in compression),

$$\varepsilon_s' < \varepsilon_{yd} \longrightarrow \frac{\xi_d - \delta_m}{\xi_d} \cdot \varepsilon_{cu} < \varepsilon_{yd} \rightarrow \xi_d - \delta < \frac{\varepsilon_{yd} \cdot \xi_d}{\varepsilon_{cu}}$$

$$\longrightarrow \delta_{min} > \xi_d - \frac{\varepsilon_{yd}}{\varepsilon_{cu}} \cdot \xi_d$$

Let's check this minimum value of δ that ensures $\varepsilon_s' < \varepsilon_{yd}$

EXAMPLE $\quad f_{yk} = 500$ MPa, class C $\longrightarrow \varepsilon_{yd} = \frac{500/1.15}{200 \cdot 10^3} = 2.17‰$

$$\xi_d = \frac{3.5}{3.5 + 2.17} = 0.617 \longrightarrow \delta_{min} = 0.234 \longrightarrow d' > 23\% \, d$$

In most cases, ~~the~~ top steel will yield in compression!

Now, if we consider the same assumption of point C for the strain of the top steel, we get:

$$\varepsilon_s' = \frac{\delta - \xi_d}{\xi_d} \cdot \varepsilon_{cu}$$

this way, we can take ε_{cu} as the absolute value,
and $\delta - \xi_d$, being < 0, will give a negative value of ε_s'

Therefore, $\sigma_s' = E_s \cdot \varepsilon_s'$ (if $|\varepsilon_s'| \leq |\varepsilon_{yd}|$!) is less than 0, that is, compression stress is negative

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Now, let's assume
$$\sigma_s' = \begin{cases} E_s \cdot \varepsilon_s' & \text{if } \delta > \xi_d - \dfrac{\varepsilon_{yd}}{\varepsilon_{cu}} \cdot \xi_d \\ -f_{yd} & \text{otherwise} \end{cases}$$

with $\varepsilon_s' = \dfrac{\delta - \xi_d}{\xi_d} \cdot \varepsilon_{cu}$ → negative in compression

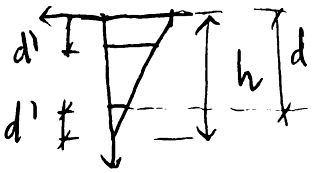
$$\boxed{N_{Rd,D}} = 0.8\, \xi_d \cdot d \cdot b \cdot f_{cd} - f_{yd} A_s - \sigma_s' A_s'$$

positive contribution if $\sigma_s' < 0$ → compression!

$$\boxed{M_{Rd,D}} = 0.8\, \xi_d\, d\, b\, f_{cd} \left(\frac{h}{2} - 0.4\, \xi_d \cdot d\right) + \left(f_{yd} \cdot A_s - \sigma_s' A_s'\right)\left(\frac{h}{2} - d'\right)$$

again, positive contrib. if $\sigma_s' < 0$!

5) Point E (related to line e)



We have to calculate the strain of top and bottom steel reinforcements

$$\frac{\varepsilon_s'}{h-d'} = \frac{\varepsilon_{cu}}{h} \rightarrow \varepsilon_s' = \frac{\overset{d+d'}{h-d'}}{\underset{d+d'}{h}}\, \varepsilon_{cu} = \frac{\cancel{d}\,(1 + \cancel{d'/d} + \cancel{d'/d})}{\cancel{d}\,(1 + \underbrace{d'/d}_{\delta})}\, \varepsilon_{cu}$$

$$\rightarrow \varepsilon_s' = \frac{1}{1+\delta}\, \varepsilon_{cu}$$

Could be $\varepsilon_s' < \varepsilon_{yd}$, which means top steel in elastic range?

$$\varepsilon_s' < \varepsilon_{yd} \rightarrow \frac{1}{1+\delta}\, \varepsilon_{cu} < \varepsilon_{yd} \rightarrow 1 + \delta > \frac{\varepsilon_{cu}}{\varepsilon_{yd}} \rightarrow \delta > \frac{\varepsilon_{cu}}{\varepsilon_{yd}} - 1$$

As before, let consider $\varepsilon_s' = -\dfrac{1}{1+\delta}\, \varepsilon_{cu}$ → absolute value

to take into account that $\sigma_s' < 0$ → compression stress

Now, look at the bottom steel

$$\frac{\varepsilon_s}{d'} = \frac{\varepsilon_{cu}}{h} \rightarrow \varepsilon_s = \frac{d'}{d+d'}\, \varepsilon_{cu} \rightarrow \varepsilon_s = \frac{\delta}{1+\delta}\, \varepsilon_{cu}$$

Could be $\varepsilon_s < \varepsilon_{yd}$? $\longrightarrow \cancel{(1+\delta)\varepsilon}\ \dfrac{\delta}{1+\delta} \cdot \varepsilon_{cu} < \varepsilon_{yd}$

$$(1+\delta)\, \varepsilon_{yd} > \delta\, \varepsilon_{cu} \rightarrow \left(\frac{1}{\delta} + 1\right) > \frac{\varepsilon_{cu}}{\varepsilon_{yd}} \rightarrow \frac{1}{\delta} > \frac{\varepsilon_{cu}}{\varepsilon_{yd}} - 1$$

$$\rightarrow \frac{1}{\delta} > \frac{\varepsilon_{cu} - \varepsilon_{yd}}{\varepsilon_{yd}} \rightarrow \delta < \frac{\varepsilon_{yd}}{\varepsilon_{cu} - \varepsilon_{yd}}$$

again, let's assume $\varepsilon_s = -\frac{\delta}{1+\delta}\varepsilon_{cu}$ in order to get negative strain in compression

Stress in steel reinforcements: σ_s' (top) and σ_s (bottom), with $\sigma < 0$ in compression (we just want that the force, not the ~~tension~~ stress, be positive in compression!)

$$\sigma_s' = \begin{cases} E_s \cdot \varepsilon_s' & \text{if } \delta > \frac{\varepsilon_{cu}}{\varepsilon_{yd}} - 1 \\ -f_{yd} & \text{otherwise} \end{cases}$$

absolute value

$\varepsilon_s' = -\frac{1}{1+\delta}\varepsilon_{cu}$

negative in compression

$\varepsilon_s = -\frac{\delta}{1+\delta}\varepsilon_{cu}$

$$\sigma_s = \begin{cases} E_s \cdot \varepsilon_s & \text{if } \delta < \frac{\varepsilon_{yd}}{\varepsilon_{cu} - \varepsilon_{yd}} \\ -f_{yd} & \text{otherwise} \end{cases}$$

$$\boxed{N_{Rd,E}} = 0.8\,h\,b\,f_{cd} - \sigma_s A_s - \sigma_s' A_s'$$

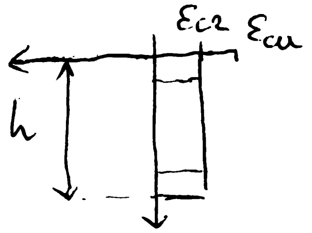
$\sigma_s, \sigma_s' < 0 \rightarrow$ positive contrib. to N!

$$\boxed{M_{Rd,E}} = \underbrace{0.8\,h\,b\,f_{cd}\left(\frac{h}{2} - 0.4h\right)}_{\text{This can be rewritten as: } 0.08\,h^2 b f_{cd}} + \left(\sigma_s A_s - \sigma_s' A_s'\right)\left(\frac{h}{2} - d'\right)$$

bottom steel in compression → negative contribution to M

top steel in compression → positive contribution to M

6) Point F (related to line f)



All the points are uniformly compressed

$\varepsilon_s = \varepsilon_s' = \varepsilon_{c2} \rightarrow$ limit strain $< \varepsilon_{cu}$ to ensure a residual deformability of the section

$$N_{Rd,F} = f_{cd} \cdot b \cdot h - \sigma_s A_s - \sigma_s' A_s'$$

← of course, $\sigma_s = \sigma_s'$ and both are compression

$$\sigma_s = \sigma_s' = \begin{cases} -E_s \cdot \varepsilon_{c2} & \text{if } |\varepsilon_{c2}| < |\varepsilon_{yd}| \\ -f_{yd} & \text{otherwise} \end{cases}$$

absolute value... and compressive stresses are negative (but compressive forces are positive, in our case!)

$$\boxed{N_{Rd,F}} = f_{cd} \cdot bh - \sigma_s (A_s + A_s')$$

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$$M_{Rd,F} = 0 + (\sigma_s A_s - \sigma_s' A_s') \cdot \left(\frac{h}{2} - d'\right)$$

Concrete compressive force is centred! Thus it does not contribute to M_{Rd}

bottom steel in tension → positive contribution to M_{Rd} (but it's compressed → negative contrib.)

top steel in tension → negative contribution to M_{Rd}. (but it's compressed, i.e. $\sigma_s' < 0$, thus its contribution is positive)

$(\sigma_s = \sigma_s')$

$$\boxed{M_{Rd,F}} = \sigma_s (A_s - A_s') \left(\frac{h}{2} - d'\right)$$

SUMMARY Now we are able, given a certain cross section (i.e. knowing b, h, d'; A_s and A_s'; type of steel and type of concrete, i.e. f_{yk} and f_{ck}), to calculate six significant points in the plane N, M (axial force – bending moment). If we connect those points with straight segments, we can get an approximation of the domain N_{Rd}, M_{Rd} (resisting ~~bendi~~ axial force and resisting bending moment) which tells us the "safe" combinations of N, M that our cross section can withstand, the limit combinations that equates the resisting forces, and the combinations outside the domain that our section cannot withstand.

NOTE We moved from other simplifications, like the adoption of a stress-block material law for concrete, and the adoption of a horizontal branch (with no hardening) for steel.

EXAMPLE Let's consider a section 30×50 cm, with $4\phi 20$ in the bottom side and $2\phi 20$ in the top side. The cover d' is equal to 5 cm.
Now we want to calculate its interaction diagram (N, M), using concrete C30/37 and steel with $f_{yk} = 500$ MPa, class C

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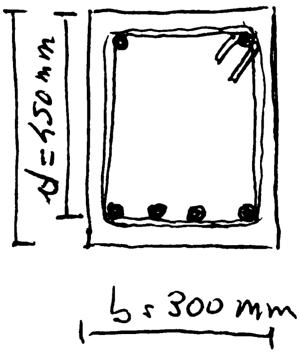
$h = 500$ mm



$d' = 50$ mm

$d' = 50$ mm

$b = 300$ mm

$\delta = \frac{d'}{d} = 0.111$

Concrete C30/37

$f_{ck} = 30$ MPa

$f_{cd} = \frac{0.85 \cdot f_{ck}}{1.50} \cong 0.567 f_{ck} \overset{!}{=} 17$ MPa

$\varepsilon_{cu} = 3.5$ ‰

$\varepsilon_{c2} = 2$ ‰

STEEL $f_{yk} = 500$ MPa, $E_s = 200$ GPa

class C $\longrightarrow \varepsilon_{uk} = 7.5\% \Rightarrow \varepsilon_{ud} = 0.9\,\varepsilon_{uk} = 67.5$ ‰

$A_s = 4 \cdot \frac{20^2}{4}\pi = 1257\ mm^2$

$A_s' = 628\ mm^2$

$A_s'/A_s = 0.5$

$f_{yd} = \frac{f_{yk}}{1.15} = 435$ MPa

$\varepsilon_{yd} = \frac{f_{yd}}{E_s} = 2.18$ ‰ ($\cong 62.3\%$ of ε_{cu})

1) POINT A

$N_{Rd,A} = -f_{yd}(A_s + A_s') = -435 \frac{N}{mm^2} \cdot 1.5 \cdot 1257\ mm^2$

$\overset{!}{=} -820.2\ kN$ (tensile force $\rightarrow$ negative for this problem only)

$M_{Rd,A} = f_{yd}(A_s - A_s')\left(\frac{h}{2} - d'\right) = 435\,(0.5 \cdot 1257)\,\underbrace{\left(\frac{500}{2} - 50\right)}_{200\ mm}$

N/mm^2 $\qquad mm^2$

$\overset{!}{=} 54.68 \cdot 10^6$ Nmm

$\overset{!}{=} 54.68$ kNm

kN $\qquad$ kN·m

A = (−820.2; 54.68)

2) POINT B

$\varepsilon_s' = \frac{d'}{d} \cdot \varepsilon_{ud} = 0.111 \cdot 67.5$ ‰ $= 7.49$ ‰ $> \varepsilon_{yd}$

top steel yielded in tension

B ≡ A

$N_{Rd,B} = -f_{yd}(A_s + A_s') = N_{Rd,A} = -820.2$ kN

$M_{Rd,B} = M_{Rd,A} = 54.68$ kN·m

3) POINT C

$\varepsilon_s' = \frac{\delta - \xi_c}{\xi_c} \cdot \varepsilon_{cu}$

$\overset{!}{=} \frac{0.111 - 0.0493}{0.0493} \varepsilon_{cu}$

$\overset{!}{=} 1.252\,\varepsilon_{cu} = 4.38$ ‰ $> \varepsilon_{yd} \rightarrow$ top steel yielded in tension

$\xi_c = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_{ud}} = \frac{3.5}{3.5 + 67.5}$

$\overset{!}{=} 0.0493$

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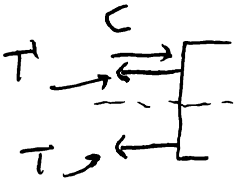

$$N_{Rd,C} = 0.8 \cdot \xi_c d b f_{cd} - f_{yd} A_s \underbrace{- f_{yd} A_s'}$$

negative because also the top steel is in tension

$$= 0.8\, \xi_c d b f_{cd} - f_{yd}(A_s + A_s')$$

$$= 0.8 \cdot 0.0493 \cdot (450\,\cancel{mm} \cdot 300\,\cancel{mm}) \cdot 17 \frac{N}{\cancel{mm^2}} + $$

$$- 435 \frac{N}{\cancel{mm^2}} \cdot 1.5 \cdot 1257\,\cancel{mm^2} =$$

$$= 90515 + (-820193)\,N = -729.7\ kN$$

$$M_{Rd,C} = 0.8 \cdot \xi_c d b f_{cd} \left(\frac{h}{2} - 0.4\, \xi_c d\right) +$$

$$+ (f_{yd} A_s - \sigma_s' A_s')\left(\frac{h}{2} - d'\right)$$

$\sigma_s' \to f_{yd}$ because $\varepsilon_s' > \varepsilon_{yd}$

kN, kN·m

$$C = (-729.7;\ 76.51)$$

$$= 0.8 \cdot 0.0493 \cdot 450 \cdot 300 \cdot 17 \cdot (250 - 0.4 \cdot 0.0493 \cdot 450)$$

$$+ 435\,(0.5 \cdot 1257)(250 - 50) \quad N \cdot mm$$

$$= 21.83 + 54.68\ kN \cdot m = 76.51\ kN \cdot m$$

241.1

4) POINT D

$$\xi_d = \frac{\varepsilon_{cu}}{\varepsilon_{cu} + \varepsilon_{yd}} = \frac{3.5}{3.5 + 2.18} = 0.6162$$

$$\varepsilon_s' = \frac{\xi_d - \delta}{\xi_d} \cdot \varepsilon_{cu} = \frac{0.6162 - 0.1111}{0.6162} \cdot 3.5 = 2.87\ ‰$$

$\varepsilon_s' > \varepsilon_{yd} \longrightarrow$ top steel yielded in compression

$\sigma_s' = -f_{yd}$ ($\sigma < 0$ in compression)

kN, kN·m

$$D = (857.9;\ 321.42)$$

$$N_{Rd,D} = 0.8 \cdot \xi_d \cdot b d \cdot f_{cd} - f_{yd} A_s + f_{yd} A_s'$$

$-(-f_{yd})$

$$= 0.8 \cdot 0.6162 \cdot 300 \cdot 450 \cdot 17 + 435\,(-0.5 \cdot 1257) \quad N$$

$$= \underbrace{1131.3}_{C} - 273.4\ kN = 857.9\ kN$$

$$M_{Rd} = \overbrace{0.8 \cdot \xi_d b d f_{cd}}^{1131.3\ kN} \cdot \left(\frac{h}{2} - 0.4\, \xi_d \cdot d\right) + (f_{yd} A_s - \sigma_s' A_s')\left(\frac{h}{2} - d'\right)$$

$\sigma_s' \to -f_{yd}$

$$= \cancel{1131.3} \cdot 157.35 + 164.07\ kN \cdot m$$

$$= 321.42\ kN \cdot m$$

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5) Point E

$$\varepsilon_s' = \frac{h-d'}{h} \cdot \varepsilon_{cu} = \frac{500-50}{500} \cdot 3.5 \overset{!}{=} 3.15 ‰ > \varepsilon_{yd}$$

top steel yielded in compression

$$\varepsilon_s = \frac{50}{500} \varepsilon_{cu} = 0.35 ‰$$

bottom steel in the elastic range (in compression too)

$$\sigma_s' = -f_{yd}, \quad \sigma_s = -E_s \cdot |\varepsilon_s|$$

$$N_{Rd,E} = 0.8\, h b f_{cd} - \sigma_s A_s - \sigma_s' A_s'$$
$$\overset{!}{=} 0.8\, h b f_{cd} - (-E_s |\varepsilon_s|) A_s - (-f_{yd} A_s')$$
$$\overset{!}{=} 0.8\, h b f_{cd} + E_s |\varepsilon_s| A_s + f_{yd} A_s'$$
$$= 0.8 \cdot 500 \cdot 300 \cdot 17 + 200 \cdot 10^3 \cdot 0.35‰ \cdot 1257 + 435 \cdot 628 \text{ N}$$
$$= 2040 \text{ kN} + 87.99 \text{ kN} + 273.18 \text{ kN}$$
$$\overset{!}{=} 2401.2 \text{ kN}$$

kN, kN·m

$$(E) = (2401.2, 139.04)$$

$$M_{Rd} = 0.08\, h^2 b f_{cd} + (-E_s |\varepsilon_s| A_s - (-f_{yd}) \cdot A_s')\left(\frac{h}{2} - d'\right)$$
$$\overset{!}{=} 0.08\,(0.5^2) \cdot 0.3 \cdot \underbrace{17 \cdot 10^3}_{f_{cd} \text{ in kN/m}^2} + (-\underbrace{200 \cdot 10^6}_{E_s \text{ in kN/m}^2} \cdot 0.35 \cdot 10^{-3} \cdot \underbrace{1257 \cdot 10^{-6}}_{A_s \text{ in m}^2} + 435 \cdot 10^3 \cdot 628 \cdot 10^{-6}) \, (0.25 - 0.05)$$
$$\overset{!}{=} 102 + (-87.99 + 273.18) \cdot 0.20 \text{ kN·m}$$
$$\overset{!}{=} 102 + 37.04 = 139.04 \text{ kN·m}$$

6) POINT F

$$\varepsilon_s = \varepsilon_s' = \varepsilon_{c2} = 2 ‰ < \varepsilon_{yd}$$

top and bottom steel within the elastic range

$$\sigma_s = \sigma_s' = -E_s \cdot \varepsilon_{c2} = -400 \text{ MPa}$$

$$N_{Rd,F} = b h f_{cd} - \sigma_s (A_s + A_s') \qquad \sigma_s < 0 \,!$$
$$\overset{!}{=} 0.3 \cdot 0.5 \cdot 17 \cdot 10^3 - (-400 \cdot 10^3)(1.5 \cdot 1257 \cdot 10^{-6}) \text{ kN}$$
$$\overset{!}{=} 2550 \text{ kN} + 754.2 \text{ kN} = 3304.2 \text{ kN}$$

$$(F) = (3304.2; -50.28)$$

$$M_{Rd,F} = \sigma_s (A_s - A_s')\left(\frac{h}{2} - d'\right) = -400 \cdot 10^3 (0.5 \cdot 1257 \cdot 10^{-6}) \cdot 0.2$$
$$\overset{!}{=} -50.28 \text{ kN m}$$

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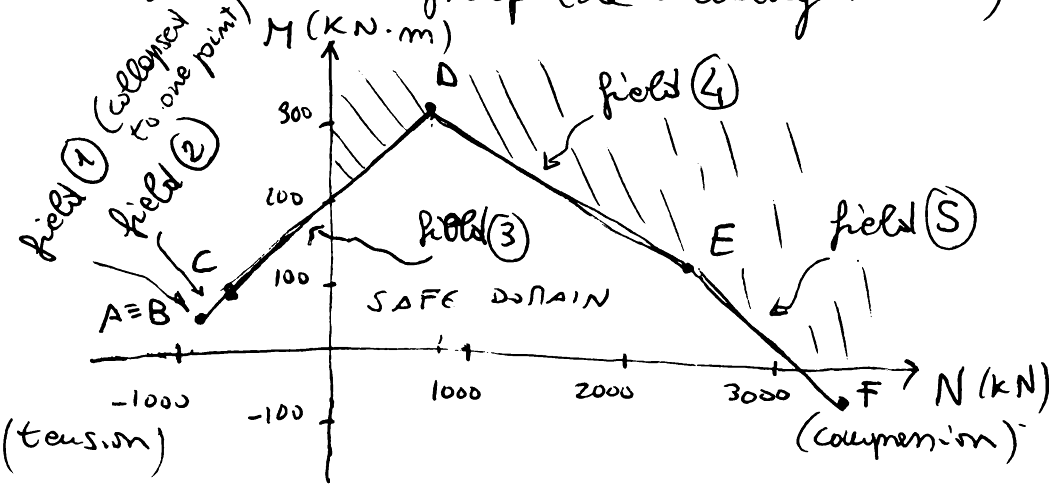
With points (A), (B), (C), (D), (E) and (F) we get half of the diagram N-M, and it would be usually enough because it is complete in the quadrant $N>0$ (compression) and $M>0$. If you want to find the remaining part of the interaction diagram, you have just to reverse the position of top and bottom steel (and to adjust the sign of the bending moment).

A (−820.2; 54.68)
B ≡ A
C = (−729.7; 76.51)
D (857.9; 321.42)
E (2401.2; 139.4)
F (3304.2; −50.28)

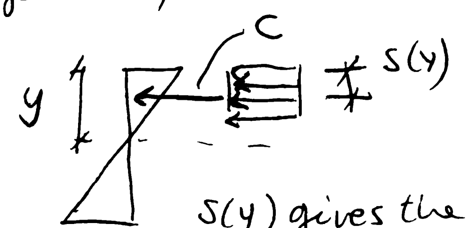


In practice, you would prefer to use more general diagrams, since the one we have found depends upon: sizes of the cross section, and cover (b, h, d'); type of concrete (f_{cd}); type of steel (f_{yk}); amount of bottom and top steel (A_s, A_s').

In general, we have that:

$$N_{Rd} = C - T' - T$$

compression is positive; C: force in concrete; T': tension in top steel; T: tension in bottom steel



s(y) gives the position of the compressive force in the concrete

$$M_{Rd} = C\left(\frac{h}{2} - s(\xi)\cdot y\right) + (T_s - T_s')\left(\frac{h}{2} - d'\right)$$

$$y = \xi d \rightarrow f(y) = f(\xi)$$

therefore, $C = \alpha(\xi)\cdot \underbrace{\xi\cdot d}_{y}\cdot b\cdot f_{cd}$, with $\alpha(\xi)$ = certain function related to the position of the neutral axis

$$T_s = \sigma_s \cdot A_s$$

$$T_s' = \sigma_s' \cdot A_s'$$

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$$\begin{cases} N_{Rd} = \alpha(\xi)\cdot\xi\, d b f_{cd} - \sigma_s' A_s' - \sigma_s A_s \\ M_{Rd} = \alpha(\xi)\,\xi\, d b f_{cd}\left(\frac{h}{2} - s(\xi)\cdot\xi d\right) + (\sigma_s A_s - \sigma_s' A_s')\left(\frac{h}{2} - d'\right) \end{cases}$$

So, N_{Rd} and M_{Rd} are functions of : ξ; b, h, d'; A_s, A_s'; f_{cd}, f_{yd}

Now, we want to reduce those parameters, so we can introduce certain quantities:

Dimensionless forces (or nominal forces) $\quad n_{Rd} = \dfrac{N_{Rd}}{b\cdot h\cdot f_{cd}}$; $m_{Rd} = \dfrac{M_{Rd}}{b\cdot h^2\cdot f_{cd}}$

Mechanical percentage of reinforcement $\quad \omega = \dfrac{f_{yd}\cdot A_s}{f_{cd}\cdot b\cdot d}$; $\omega' = \dfrac{f_{yd}\cdot A_s'}{f_{cd}\cdot b\cdot d}$

bottom steel — top steel

Reinforcement ratio $\quad \mu = \dfrac{A_s'}{A_s} \longrightarrow \omega' = \mu\cdot\omega$

Dimensionless stress of the reinforcement $\quad \eta = \dfrac{\sigma_s}{f_{yd}}$; $\eta' = \dfrac{\sigma_s'}{f_{yd}}$

bottom steel — top steel

Dimensionless cover $\quad \delta = d'/d$

Let consider $n_{Rd} = \dfrac{N_{Rd}}{bh\cdot f_{cd}} = \dfrac{\alpha(\xi)\cdot\xi\, d \cancel{b} \cancel{f_{cd}}}{\cancel{b} h \cancel{f_{cd}}} - \dfrac{\sigma_s' A_s'}{bh f_{cd}} - \dfrac{\sigma_s A_s}{bh f_{cd}}$

$$= \alpha(\xi)\cdot\xi\, d/h - \frac{\eta'\cdot f_{yd}\cdot A_s'}{bh f_{cd}} - \frac{\eta f_{yd} A_s}{bh f_{cd}}$$

[$f_{yd} A_s = \omega f_{cd} b d$]
[$f_{yd} A_s' = \omega' f_{cd} b d \overset{!}{=} \mu\omega f_{cd} b d$]

$$= \alpha(\xi)\cdot\xi\cdot\frac{d}{d+d'} - \eta'\cdot\frac{\mu\omega \cancel{f_{cd}} \cancel{b} d}{\cancel{b} h \cancel{f_{cd}}} - \eta\frac{\omega \cancel{f_{cd}} \cancel{b} d}{\cancel{b} h \cancel{f_{cd}}}$$

$$= \alpha(\xi)\cdot\xi\cdot\frac{\cancel{d}}{\cancel{d}(1+\delta)} - \eta'\mu\omega\cdot\frac{\cancel{d}}{\cancel{d}(1+\delta)} - \eta\omega\frac{\cancel{d}}{\cancel{d}(1+\delta)}$$

$$\rightarrow n_{Rd} = \frac{1}{1+\delta}\left(\alpha(\xi)\cdot\xi - \mu\eta'\omega - \eta\omega\right)$$

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Let consider $m_{Rd} = \frac{M_{Rd}}{bh^2 f_{cd}} = \alpha(\xi)\cdot \xi d b f_{cd}\left(\frac{h}{2} - s(\xi)\cdot \xi d\right)\cdot \frac{1}{bh^2 f_{cd}} +$

$$+ (\sigma_s A_s - \sigma_s' A_s')\left(\frac{h}{2} - d'\right)\cdot \frac{1}{bh^2 f_{cd}}$$

$$m_{Rd} = \alpha(\xi)\cdot \xi \cdot d \cdot \cancel{b} \cancel{f_{cd}} \cdot \frac{1}{2}\left(h - 2s(\xi)\,\xi\cdot d\right)\cdot \frac{1}{\cancel{b}h^2 \cancel{f_{cd}}} +$$

$$+ \Big(\underbrace{\eta\cdot \cancel{f_{yd}}}_{\sigma_s}\cdot \underbrace{\omega\cdot\frac{\cancel{b}d\cancel{f_{cd}}}{\cancel{f_{yd}}}}_{A_s} - \underbrace{\eta'\cdot\cancel{f_{yd}}}_{\sigma_s'}\cdot \underbrace{\mu\cdot\omega\frac{\cancel{b}d\cancel{f_{cd}}}{\cancel{f_{yd}}}}_{A_s'}\Big)\Big(\underbrace{\frac{d+d'}{2}}_{h/2} - d'\Big)\cdot \frac{1}{\cancel{b}h^2\cancel{f_{cd}}}$$

$$= \alpha(\xi)\cdot\xi\cdot\frac{d}{h}\cdot\frac{1}{2}\left(\frac{h}{h} - 2s(\xi)\cdot\xi\cdot\frac{d}{h}\right) + \omega(\eta - \mu\eta')\cdot\frac{d}{h}\cdot\frac{d-d'}{2}\cdot\frac{1}{h}$$

h^2

$$= \alpha(\xi)\cdot\xi\left(\frac{d}{d+d'}\right)\cdot\frac{1}{2}\left(1 - 2s(\xi)\cdot\xi\cdot\frac{d}{d+d'}\right) + \omega(\eta - \mu\eta')\cdot\frac{d}{d+d'}\cdot\frac{d-d'}{d+d'}\cdot\frac{1}{2}$$

$\frac{\cancel{d}}{\cancel{d}(1+\delta)}$

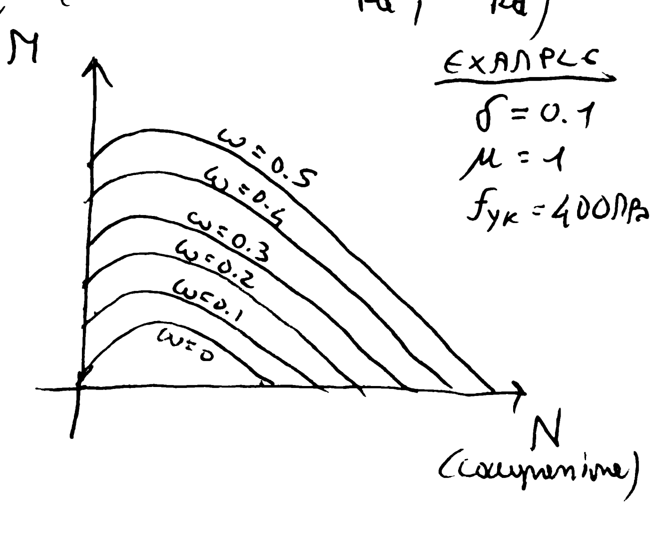
$$= \alpha(\xi)\cdot\xi\cdot\frac{1}{1+\delta}\cdot\frac{1}{2}\left(\frac{1+\delta}{1+\delta} - 2s(\xi)\cdot\xi\cdot\frac{1}{1+\delta}\right) + \omega(\eta-\mu\eta')\cdot\frac{1}{1+\delta}\cdot\frac{\cancel{d}(1-\delta)}{\cancel{d}(1+\delta)}\cdot\frac{1}{2}$$

$$= \frac{1}{(1+\delta)^2}\left[\alpha(\xi)\cdot\xi\cdot\left(\frac{1+\delta}{2} - s(\xi)\cdot\xi\right) + \omega\,\frac{1-\delta}{2}(\eta - \mu\eta')\right]$$

Looking at the expression for n_{Rd} and m_{Rd}, given a certain cross section, we know: δ (related to the cover), μ (ratio between top and bottom reinforcement), ω (amount of reinforcement), type of steel given by f_{yk} (which influences the position of some of the limit curves)

The independent variable of our problem is $\xi \in]-\infty; +\infty[$, which gives $\alpha(\xi)$, $s(\xi)$, η and η' (and thus n_{Rd}, m_{Rd})

For the design of our cross section, we can use diagrams provided for certain combinations of μ, δ and f_{yk}. Each diagram contains several curves related to increasing amounts of ω



EXAMPLE
$\delta = 0.1$
$\mu = 1$
$f_{yk} = 400\,MPa$