LECTURE 6

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Recap. In the previous lecture, we saw how to face the problem of evaluating the capacity of a rectangular cross section subjected to combined bending and axial loading. The procedure allows deriving interaction diagrams that link the limit combinations of axial force (N) and bending moment (M) that a certain section can withstand. In other words, the curve marks the resisting values of internal forces (N_{Rd}, M_{Rd}), while the portion of the plane included by that closed and convex domain indicates the "safe" combinations of (N, M), which our section can withstand without reaching a failure condition. Points outside the domain are not allowed.

Moreover, we saw a simplified procedure to obtain an approximated diagram, based on the property of convexity, which means that, if we connect with straight lines some points belonging to the actual curve, we get a polyline that is included in the refined diagram. Of course, the chosen points correspond to the limit lines that were related to the various failure condition of the cross section, like the maximum strain of steel, or the maximum strain of concrete.

To design a certain cross section (i.e. to decide the amount of steel required to resist to the design values of axial force and bending moment), usually you can use pre-built diagrams. As we saw, a certain (N_{Rd}, M_{Rd}) domain is depending upon: the sizes of the cross section (b, h and d'), cover included; the strength of concrete (f_{cd}); the strength of steel (f_{yd}); the amount of bottom and top steel (A_s and A'_s), which can be related by the ratio $u = A'_s/A_s$.

In order to get more general diagrams, usually the internal forces get normalized with respect to $b \cdot h \cdot f_{cd}$ (axial force) and $b \cdot h^2 \cdot f_{cd}$ (bending moment), thus obtaining charts about $n = N/bhf_{cd}$ and $m = M/bh^2f_{cd}$, which are called dimensionless (or relative) axial force and bending moment, respectively.

Moreover, each chart contains several curves related to increasing amounts of steel, usually varying according to predefined steps expressed by the mechanical percentage of reinforcement $\omega = \frac{A_s \cdot f_{yd}}{bd\, f_{cd}}$ (example: $\omega = 0, 0.1, 0.2, \ldots, 1$)
However, each single chart (which contains a set of curves) is related to a certain value of the cover, usually expressed by the dimensionless quantity $\delta = d'/d$, which influences the lever arms of the internal forces; is related to a certain ratio of top/bottom reinforcement μ (if $\mu = 1$, therefore the amount of steel is symmetrically placed); and, finally, is related to a certain value of the yielding strength f_{yd} of the steel. Conversely, each chart does not depend anymore upon the cross sectional sizes b, h, and upon the concrete strength f_{cd}.
Pay attention that different handbooks and different authors use various parameters, and all the required quantities should be double-checked.
For example, you could find diagrams that omit the information about μ: they maybe consider only symmetrical sections ($\mu = 1$), which is a very frequent case, but you should check if it is so. In other cases, the cover could be expressed as the ratio d'/h instead of d'/d.
The amount of steel used to derive ω could be the total amount of reinforcement (bottom + top).
Moreover, characteristic values (f_{ck}, f_{yk}) could be used, instead of design values (f_{cd}, f_{yd}).

- The last remark is related to this: the procedure to calculate the failure condition is based only on the property of the cross section, and does not consider the global behaviour of the structure element. It relies only on the first order (linear) analysis carried out to derive the design internal forces, based on the initial (not deformed) shapes of structure members.

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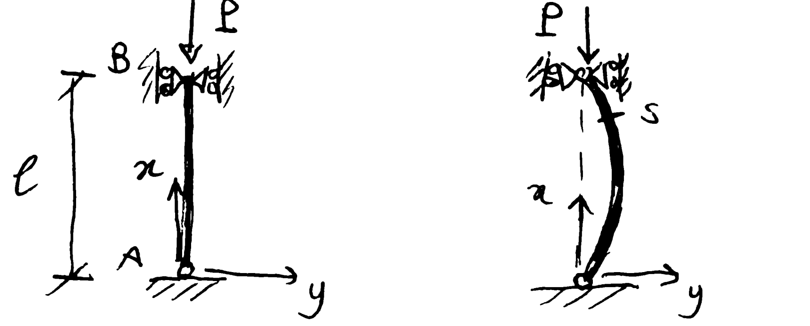
Thus, this kind of analysis does not account for any problem of instability, and can be applied only to stocky, short members.
Otherwise, in the case of slender columns, the problem of deflection, which enhances the internal forces, has to be properly taken into account, in order to avoid a possible buckling.

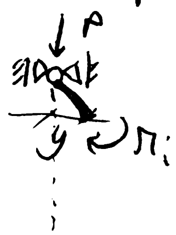
- INSTABILITY OF SLENDER COLUMNS

The basic model accounting for the problem of instability of columns is that of Euler. It considers a column with a double hinge at the ends, subjected to a compression axial load. The beam is made of an elastic material having a Young's modulus of E, and I is the second moment of area (moment of inertia) of the cross section.



initial shape   deformed shape

Differently from a first order analysis, when external forces, reactions and internal forces are referred to the initial undeformed shape, now we want to consider a possible equilibrium in a deformed situation.



Let consider the generic cross section S:
If you calculate the equilibrium with respect to the barycentre of the cross section, you can calculate the moment due to the external force P, that is $M_e = P \cdot y$. The internal moment due to the deformation of the member is $M_i = EI \cdot \chi$

M_i — under the hypothesis of elasticity; EI — bending stiffness; χ — curvature

Of course, the external moment M_e makes the beam get away from the initial shape, while the internal moment M_i tends to bring back the member to the straight position.

If $M_e > M_i$, the beam goes away from the straight line and buckling occurs.

If $M_e < M_i$, the stiffness of the beam is sufficient to go back to the initial position.

If $M_e = M_i$, the situation is in equilibrium. The load related to the possibility of having other possible equilibrated configurations, other than the initial straight shape, is therefore a critical load, P_E

$P < P_E \rightarrow$ the beam persists in its initial shape.

$P > P_E \rightarrow$ the beam fails in buckling

$P = P_E \rightarrow$ critical value of load that separates stability from instability.

Evaluation of P_E $\qquad M_e = M_i \Rightarrow P \cdot y(x) = M_i(x) \overset{!}{=} (E \cdot I) \cdot \chi(x)$

curvature $$\chi(x) = -\frac{y''(x)}{(1+\underbrace{y'(x)}_{\frac{dy(x)}{dx}}{}^2)^{3/2}} \cong -\underbrace{y''(x)}_{\frac{d^2y(x)}{dx^2}}$$

$$P \cdot y(x) + EI \cdot \frac{d^2 y(x)}{dx^2} = 0$$

Be $y = y(x)$, $y'' = d^2y(x)/dx^2$, $\alpha^2 = \frac{P}{E \cdot I}$ $\quad$ (so $\alpha = \sqrt{\frac{P}{EI}}$)

$$y'' + \alpha^2 y = 0$$

general solution ($\alpha^2 > 0$!) $\rightarrow y = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)$

Let apply the boundary conditions.

1) for $x = 0$, $y = 0$ (hinge!) $\rightarrow y = c_1 \cdot \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} \overset{!}{=} c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

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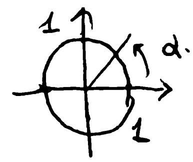
2) for $x = \ell$, $y = 0 \rightarrow y = C_1 \sin(\alpha \cdot \ell) = 0$

trivial solution : $C_1 = 0$

otherwise : $\alpha \cdot \ell = n\pi$



$$\sqrt{\frac{P_E}{EI}} \cdot \ell = n\pi$$

we can thus obtain the value of critical load $P_E = \frac{n^2\pi^2 EI}{\ell^2}$

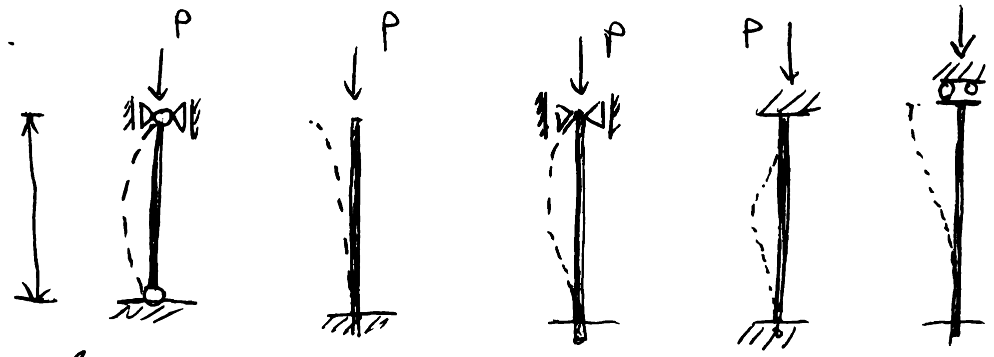
the minimum value ($n = 1$) is that we are looking for

$$\boxed{P_{crit}} = P_{E\,(n=1)} = \frac{\pi^2 EI}{\ell^2}$$

- How to generalize this result for different end restraints (other than two hinges)

We can extend this results introducing the concept of effective length ℓ_0, which can be intended as half of the wavelength that describes the deformed shape of our member.

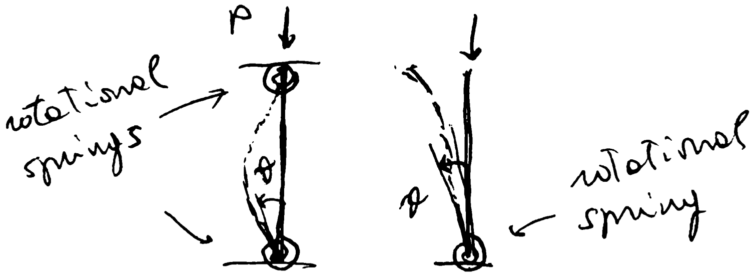
FUNDAMENTAL CASES



$\ell_0 = \ell$ $\quad \ell_0 = 2\ell$ $\quad \ell_0 = 0.7\ell$ $\quad \ell_0 = 0.5\ell$ $\quad \ell_0 = \ell$

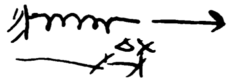
Actually, for columns belonging to ordinary frames, the situation is generally intermediate between fixed ends and perfect hinges, due to the rotational stiffness of columns and beams owning the end nodes of the frame.

The situation can be sketched in this way :



a rotational spring behaves like a spring, but in relation to moments and rotations.

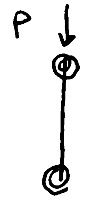
Longitudinal spring :



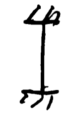
$F = k \cdot \Delta x$

Rotational spring :

$M = k \cdot \vartheta$



1 – Infinite stiffness → fixed ends, $l_0 = 0.5\,l$



2 – null stiffness → hinges, $l_0 = l$

Actual cases: $0.5\,l < l_0 < l$

1 – infinite stiffness → fixed end, $l_0 = 2\,l$

2 – null stiffness → unstable, $l_0 \to \infty$

actual cases: $l_0 > 2l$

At this point, we need a criterion to decide if instability is an issue or not.
If it is not, we can just consider the capacity of the cross section of our column, without any other global consideration that involves the whole member. Our design values of bending moment, M_{Ed}, and axial force, N_{Ed}, will be derived from a linear elastic analysis based on the initial not deformed shape of the structure.
Problems arise when our column becomes slender and slender.
Let consider three possible situations

case A, case B and case C $\qquad l_{0,A} < l_{0,B} < l_{0,C}$

$\qquad (I/A)_A > (I/A)_B > (I/A)_C$

I → second moment of area of the column's cross section

A → area of the cross section

$i = \sqrt{I/A}$ = radius of gyration

l_0 = effective length $= \beta \cdot l$ $\quad \begin{cases} l \to \text{length of the member} \\ \beta \to \text{coefficient that depends upon the boundary conditions} \end{cases}$

To compare, at the same time, information provided by boundary conditions and length of the member (l_0), and information provided by the cross section (i), we can introduce the concept of <u>slenderness λ</u>

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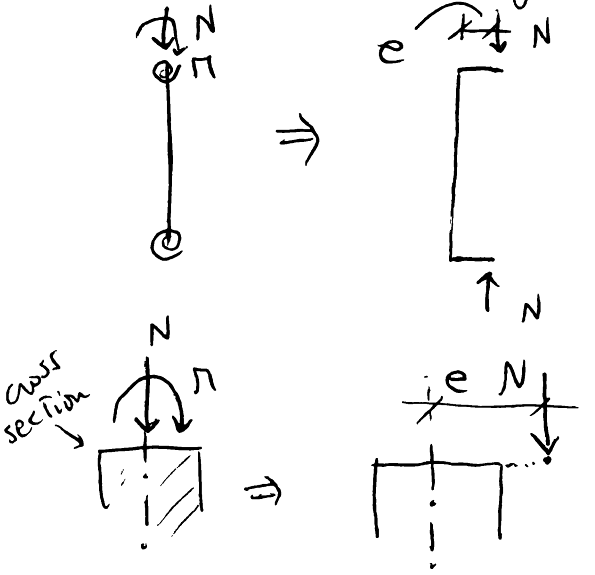
$$\lambda = \frac{l_0}{i}$$

l_0 → effective length $l_0 = \beta \cdot l$, accounting for boundary conditions and length of the column

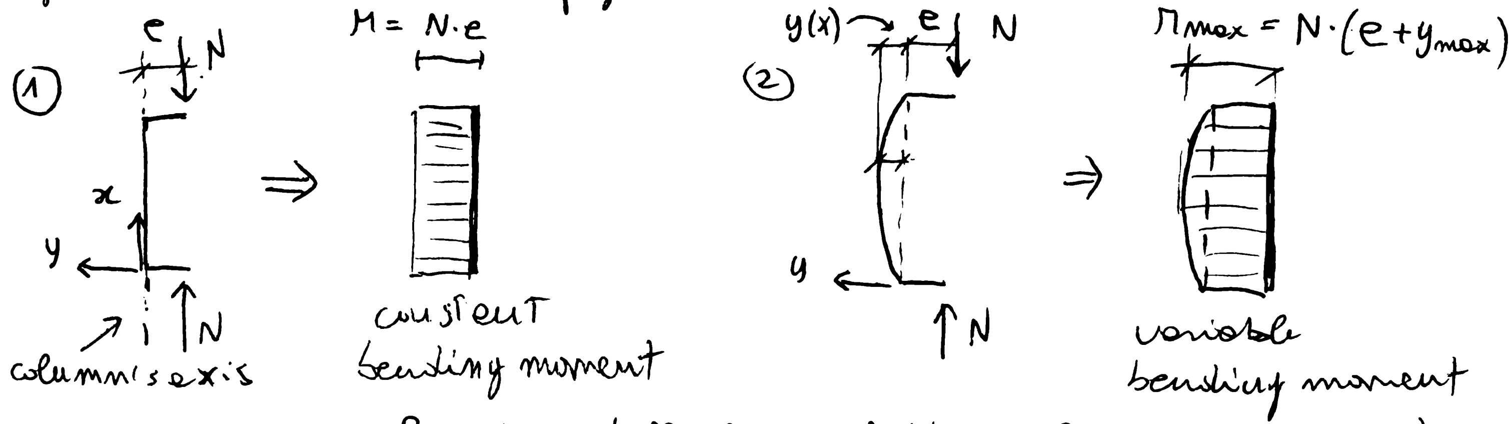
i → radius of gyration $i = \sqrt{I/A}$, accounting for the inertial properties of the cross section

In general, if λ is very low (case A), the column will behave almost linearly up to the failure due to the reach of materials strength.



We need also to introduce the concept of eccentricity e. For this kind of problem, it is generally practical to consider, instead of the internal forces N, M referred to the axis of the column, the axial force N translated of a quantity e that gives $N \cdot e = M \Rightarrow e = M/N$

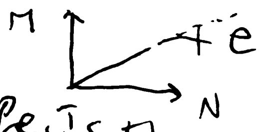
Let's consider a column subjected to an eccentric vertical load, first in the initial configuration, then in the deformed one



If we account for the deflection of the column, we find that the maximum value of bending moment is greater than the value calculated with reference to the initial undeformed shape, as usually done.
The difference is negligible (i.e. the deformation does not affect the load-carrying capacity of the column) only if the column is short and stocky, whereas the difference can be dramatic if the column becomes excessively slender.

The slenderness ratio $\lambda = l_0 / i$ represents a ~~...~~ quantity that helps describing the different behaviours of short and slender columns. There is a limit value λ_{lim} which ~~...~~ can be taken as reference, as a discrimen between the condition of being "short" and the condition of being "slender".

If $\lambda << \lambda_{lim}$, the influence of deflection is negligible, and we get a linear relation between bending moment and axial force: $M = N \cdot e$

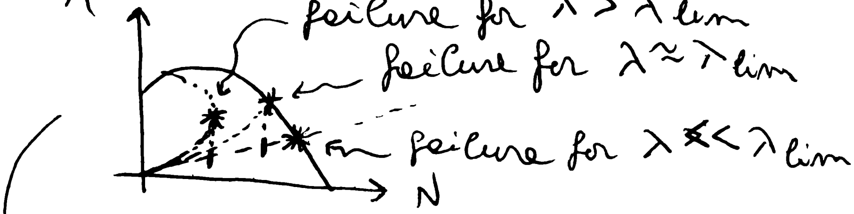


If $\lambda \simeq \lambda_{lim}$, the deflection affects the value of bending moment due to the additional eccentricity, and for increasing axial forces this effect becomes more evident

$M_{max} = N(e + y_{max})$

For $\lambda > \lambda_{lim}$, the non-linearity between M and N becomes critical.

As an example, if we combine the information about M and N (by considering, for example, an axial force that increases from 0 onward) in different cases of values of λ, together with the interaction diagrams of the cross section, we could see situations like this:



This can be summarized this way:
- for $\lambda << \lambda_{lim}$, there is a linearity between N and M and failure occurs due to the reach of materials strength (cross sectional failure);
- ~~f~~ for $\lambda \simeq \lambda_{lim}$, non-linearities become important, but however the cross sectional capacity could be reached;
- for $\lambda > \lambda_{lim}$, failure occurs before the cross-sectional failure, due to a global instability

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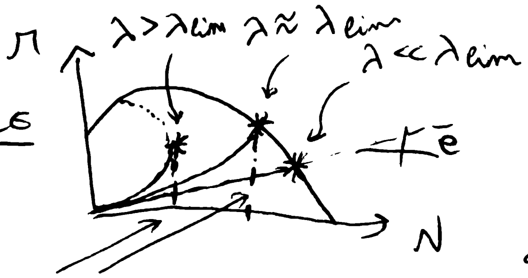
EXAMPLE



If we consider the initial shape only (~~linear~~ 1st order analysis), we do not get the actual bending moments if $\lambda \approx \lambda_{lim}$ or $\lambda > \lambda_{lim}$

First order analysis $\rightarrow M = N \cdot e$

There is an underestimation of the actual bending moment if the deflection affects the behaviour.

Basically, we can have three possible "categories" of situations:

1. $\lambda \ll \lambda_{lim} \rightarrow$ failure due to materials, cross sectional analysis sufficient to evaluate the capacity of the member;
2. $\lambda \approx \lambda_{lim} \rightarrow$ the deflection enhances the value of bending moment, and non-linearities become non negligible, however a failure due to materials only could be reached in some cases;
3. $\lambda > \lambda_{lim} \rightarrow$ exceeding lateral deflections preceed the failure due to instability, when ~~all~~ the minimum perturbation can lead to collapse (the equilibrium in the deformed situation is not stable).

From a design point of view, what we do is just the calculus of λ and the comparison to λ_{lim}:

- if $\lambda < \lambda_{lim} \Rightarrow$ design the member as a short column (interaction diagrams), just including an additional eccentricity to account for imperfections during construction
- if $\lambda \geq \lambda_{lim} \Rightarrow$ design the member as a slender column, including the ~~effe~~ 2nd order effects to increase the values of design internal forces (then, use those enhanced values for the design as short column)

Now, the point is how to calculate l_0 for members inserted into regular frames. We saw that their end restraints are something intermediate between a perfect hinge ($l_0 = l$) and a fixed restraint ($l_0 = 0.5 \cdot l$)

$$\boxed{\lambda = \frac{l_0}{i}}$$

l_0 → effective length, depending upon: actual length l; boundary conditions

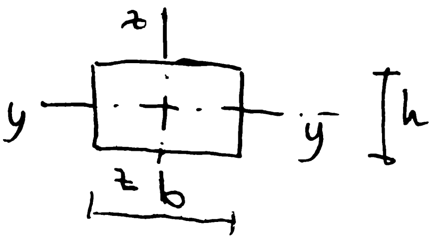
$i = \sqrt{I/A}$ (radius of gyration), with I = second moment of area (moment of inertia) of the uncracked concrete section

A = area of the cross section

Now, i can be easily calculated for a rectangular section

$I_y = \frac{1}{12} b h^3$

$[I_z = \frac{1}{12} b^3 h]$



$A = b \cdot h$

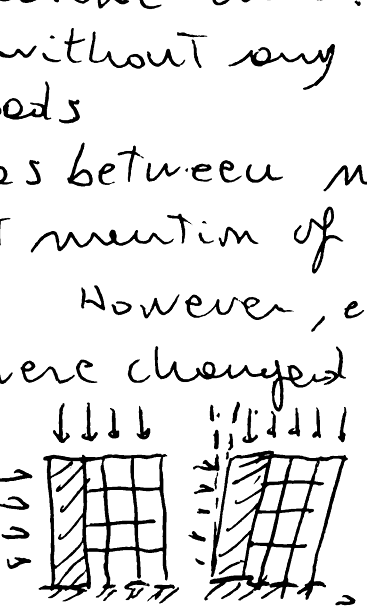
Conversely, for the calculation of l_0 we have to rely upon formulations provided by EC2.

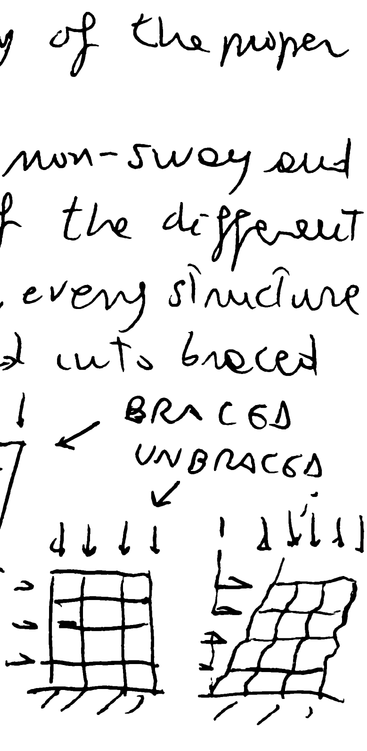
There is a difference between members that are inserted in braced frames and members belonging to unbraced frames.

BRACED FRAME: a frame where proper structural elements like shear walls, stiffening cores, or bracing systems, have been provided to resist lateral loads.

UNBRACED FRAME: a frame without any of the proper elements to resist lateral loads

(originally, the distinction was between non-sway and sway frames, with a direct mention of the different flexibility of those structures. However, every structure is deformable, so the words were changed into braced and unbraced structures)





Braced frames → stiffer, less deformable

Unbraced frames → more deformable

enhancement of non-linearities!

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Estimation of the effective length l_0

Braced members

$$l_0 = 0.5 \cdot l \cdot \sqrt{\left(1+\frac{k_1}{0.45+k_1}\right)\left(1+\frac{k_2}{0.45+k_2}\right)}$$

Unbraced members

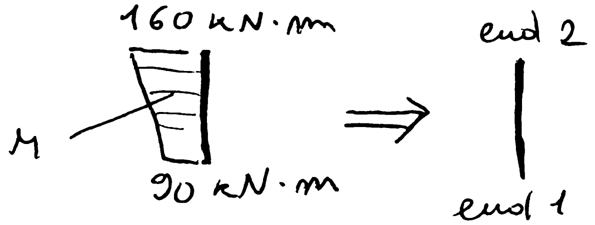
$$l_0 = \max\begin{cases} l \cdot \sqrt{1+10\,\dfrac{k_1 \cdot k_2}{k_1+k_2}} \\ l \cdot \left(1+\dfrac{k_1}{1+k_1}\right)\cdot\left(1+\dfrac{k_2}{1+k_2}\right)\end{cases}$$

l = design length (height) of the column (the actual measure)

k_1, k_2 = relative flexibilities of rotational restraints at ends 1 and 2, respectively

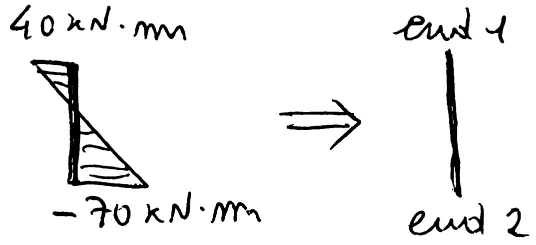
Which is end 1, and which is end 2? It depends upon the value of bending moment: $|M_2| \geq |M_1|$

example



Indeed, $|160| > |90|$!

example



indeed, $|-70| > |40|$

Rotational flexibility $k = \left(\frac{\theta}{M}\right)\cdot\left(\frac{EI}{l}\right)$

θ = rotation of restraining members for bending moment M

EI = bending stiffness of the compression member (i.e. the column)

l = clear height of the compression members between the restraints at the ends

$k = 0 \rightarrow$ null flexibility $\rightarrow$ fixed ends $\rightarrow l_0 = \begin{cases} 0.5\,l & \text{braced members} \\ l & \text{unbraced members}\end{cases}$

it means infinite rigidity!

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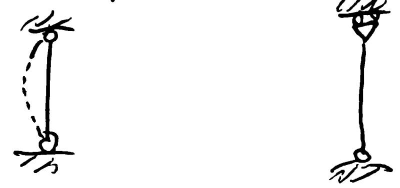
$k \to \infty \to$ infinite flexibility $\to$ hinges $\to l_0 = \begin{cases} l & \text{braced members} \\ \infty & \text{unbraced members} \end{cases}$

$l_0 = l$



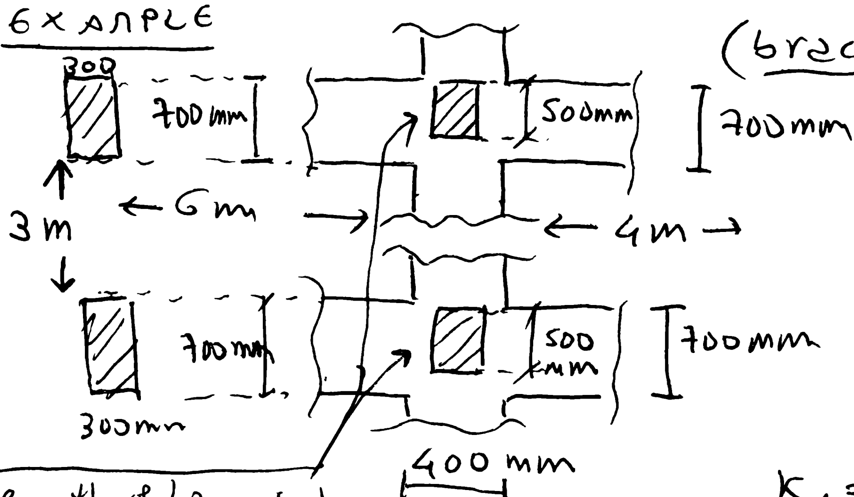
$l_0 \to \infty$ ← unstable if unbraced!

For ordinary regular frames, with spans of approximately equal length, you could assume that the adjacent columns do not contribute to the rotational stiffness of an end, and that the rotational flexibility k could be taken as:

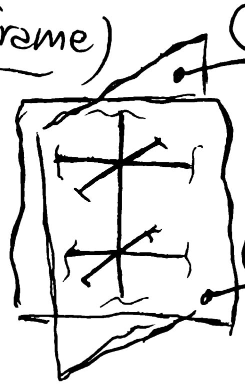
$$k = \frac{(EI/l)_{column}}{\sum_{n.beams} 2\cdot(EI/l)_{beam}} \to \left(\text{if } E \text{ is the same}\right) = \frac{(I/l)_{col}}{\sum 2(I/l)_{beam}}$$

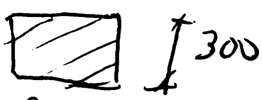
EXAMPLE

(braced frame)



length of beams 500×300 = 4 m from both sides





$k_1 = k_2 = k$ for simmetry (same elements at both sides)

Plane ①

$I_{column} = \frac{1}{12}\cdot 400^3\cdot 300 = 1.6\cdot 10^9\ mm^4$ $\quad A_{column} = 400\times 300 = 120\times 10^3\ mm^2$

$I_{beam\,(700\times 300)} = \frac{1}{12}\,700^3\cdot 300 = 8.575\cdot 10^9\ mm^4$

stiffer (lower flex!)

$$k = \frac{(\cancel{E}I/l)_{column}}{\sum 2(\cancel{E}I/l)_{beam}} = \frac{1.6\cdot 10^9/3\cdot 10^3}{(2\cdot(8.575\cdot 10^9/6\cdot 10^3 + 8.575\cdot 10^9/4\cdot 10^3))} = \boxed{0.0746}$$

Plane ②

$I_{column} = \frac{1}{12}\cdot 400\times 300^3 = 0.9\cdot 10^9\ mm^4$

more flexible!

$I_{beam\,(500\times 300)} = \frac{1}{12}\,500^3\cdot 300 = 3.125\cdot 10^9\ mm^4$

$$k = \frac{0.9\cdot 10^9/3\cdot 10^3}{2\cdot(2\cdot 3.125\cdot 10^9/4\cdot 10^3)} = \boxed{0.096}$$

k is a relative flexibility!

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Let's calculate l_0 for both planes

Plane ①

$$l_0 = 0.5 \cdot l \cdot \sqrt{\left(1 + \frac{k}{0.45+k}\right)^2} \leftarrow \begin{cases} l = 3\ m \\ k_1 = k_2 = k\ ! \end{cases}$$

$$\overset{!}{=} 0.5 \cdot (3\ m) \cdot \sqrt{\left(1 + \frac{0.0746}{0.45+0.0746}\right)^2} = 1.71\ m$$

Plane ②

$$l_0 = 0.5 \cdot l \cdot \left(1 + \frac{0.096}{0.45+0.096}\right) = 1.76\ m$$

As seen, the effective length is greater in plane 2. This could ~~have~~ had been imagined, since in plane 2 beams have a lower bending stiffness (i.e. a greater flexibility), and also the column has a lower radius of gyration.

i for plane ①: $I_{column} = 1.6 \cdot 10^9\ mm^4$, $A_{column} = 120 \times 10^3\ mm^2$

$$i = \sqrt{I/A} = \sqrt{1.6 \cdot 10^9 / 120 \cdot 10^3} = 115.5\ mm$$

i for plane ②: $I_{column} = 0.9 \cdot 10^9\ mm^4$, A_{column} is the same!

$$i = \sqrt{I/A} = \sqrt{0.9 \cdot 10^9 / 120 \cdot 10^3} = 86.6\ mm$$

Let's calculate the slenderness values for both planes.

Plane ① $\lambda = l_0 / i = \dfrac{1.71 \cdot 10^3\ mm}{115.5\ mm} = 14.8$

Plane ② $\lambda = l_0 / i = \dfrac{1.76 \cdot 10^3\ mm}{86.6\ mm} = 20.3 > 14.8\ !$