LECTURE 8

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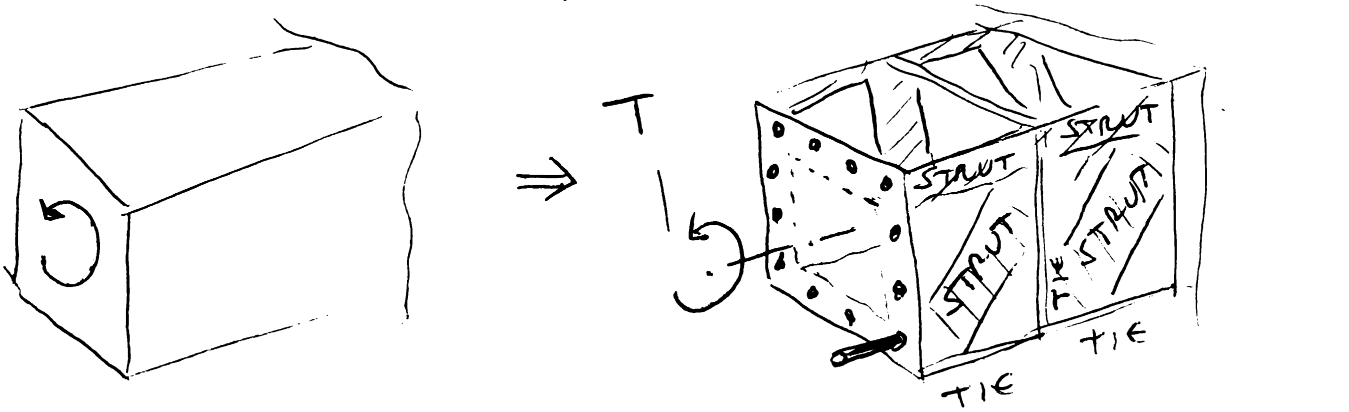
TORSION IN R.C. ELEMENTS

- A full torsional design is required only when the static equilibrium of a structure depends on the torsional resistance (primary torsion).
- Where torsion arises, in statically indeterminate structures, only due to compatibility ("congruence") and not stability, usually there is no need to check (secondary torsion). However, a minimum reinforcement should be always provided, in order to avoid excessive cracking.
- Typical situations where torsion could occur are, for example, cantilever or stairs connected to perimetric beams.
- In the following, only solid sections will be considered.
- The basic idea behind the design procedure against torsion is that the flow of tangential stresses involves only the outer parts of the section, and so the role of the inner core can be neglected.
Moreover, from a global point of view, the resisting mechanism can be described by means of an analogous truss model, similar to the one developed for shear, where a system of struts (provided by portions of concrete in compression) and ties (provided by vertical and longitudinal steel reinforcements) resist the internal force.
In the case of torsion, the truss is tridimensional, not plane like in the case of shear



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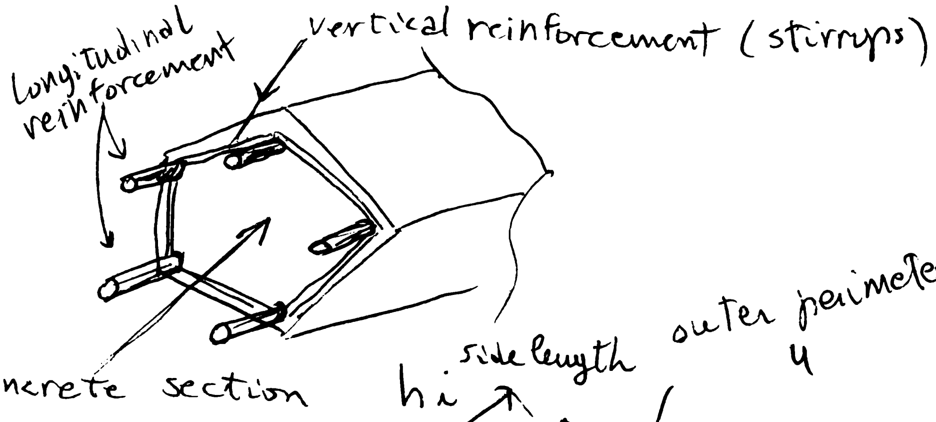
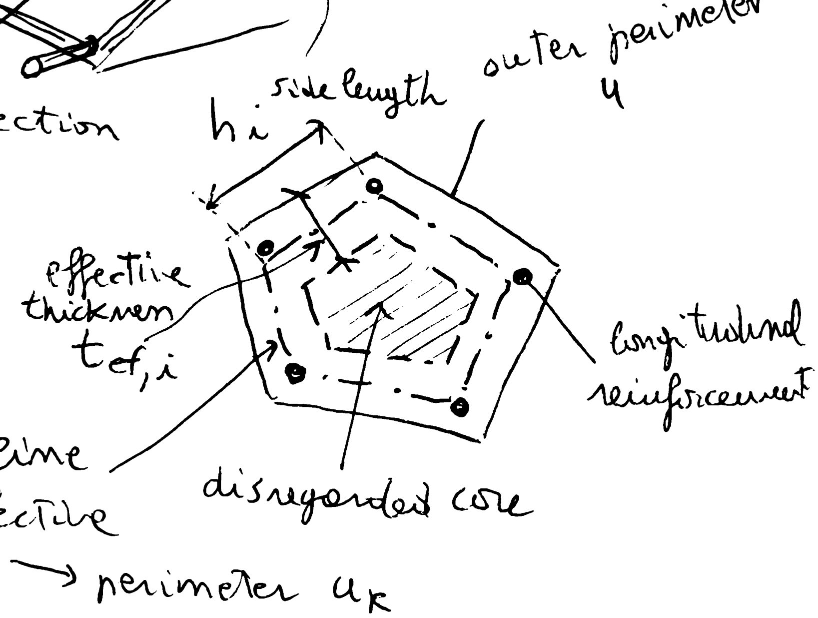
- GENERAL CROSS SECTION SUBJETED TO TORSION



As said before, the solid cross section is modelled by an equivalent hollow section, which ~~…~~ disregards the contribution of the internal core



The cross-section is then modelled by means of an effective (outer) hollow section, which ideally is effected by the tangential stresses that, once integrated, equates the torsional moment.

$$\int_{A_{eff}} \tau \, dA = T$$ torque (or torsional moment T)

That effective and ideal hollow section can be thought as an assemblage of walls, which can be studied separately

A = whole area of the cross section composed by the outer perimeter u

u = ~~…~~ outer perimeter

$t_{ef,i}$ = effective thickness of a wall of the ideal hollow effective section

h_i = side length of a wall

u_K = perimeter that stands in the middle of the effective section (i.e. its distance from the outer edge and the inner edge is equal to $\frac{1}{2} t_{ef,i}$)

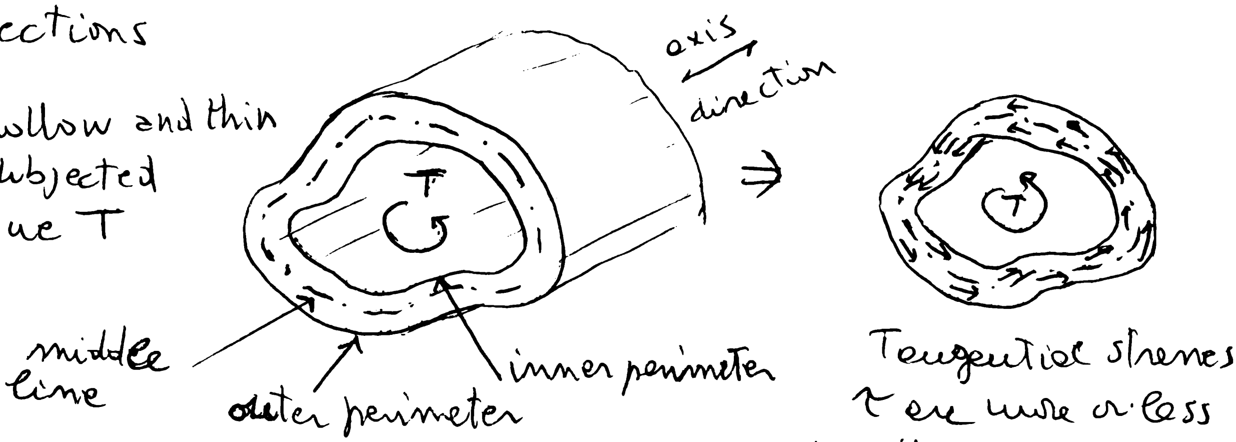
A_K = whole area enclosed by the perimeter u_K

$t_{ef} = \dfrac{A}{u}$ but it cannot be lower than 2 times the distance of the centreline of the longitudinal steel from the outer perimeter

 Once the problem has been stated, the next step is to analyze the flow of tangential stresses τ along the effective section.

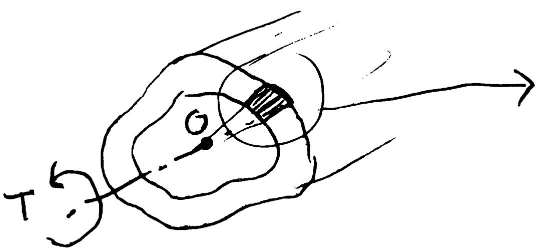
The analysis is the same developed by Bredt for thin walled sections

Generic hollow and thin section subjected to a torque T

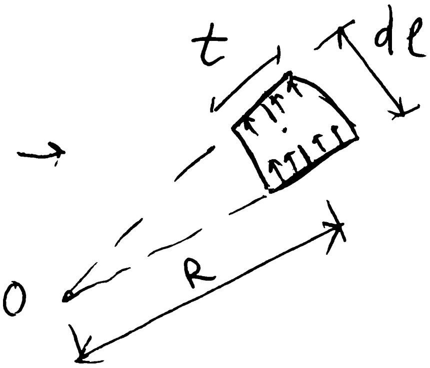


Tangential stresses τ are more or less ~~parallel~~ tangent to the perimeter

Locally, tangential stresses are tangent to the perimeter and can be considered constant through the thickness.



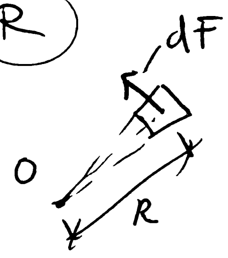
if we magnify a certain portion of the cross section (indefinitely small)



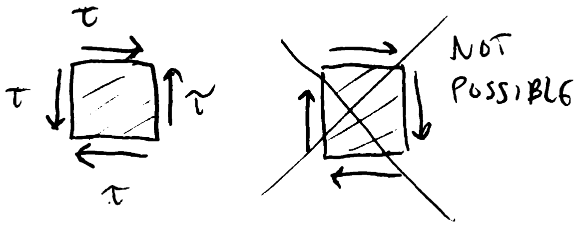
The infinitesimal force acting on this portion of area dA is dF, equal to the integral of the tangential stress in dA

$$dF = \int_{dA} \tau \cdot t \cdot dl$$

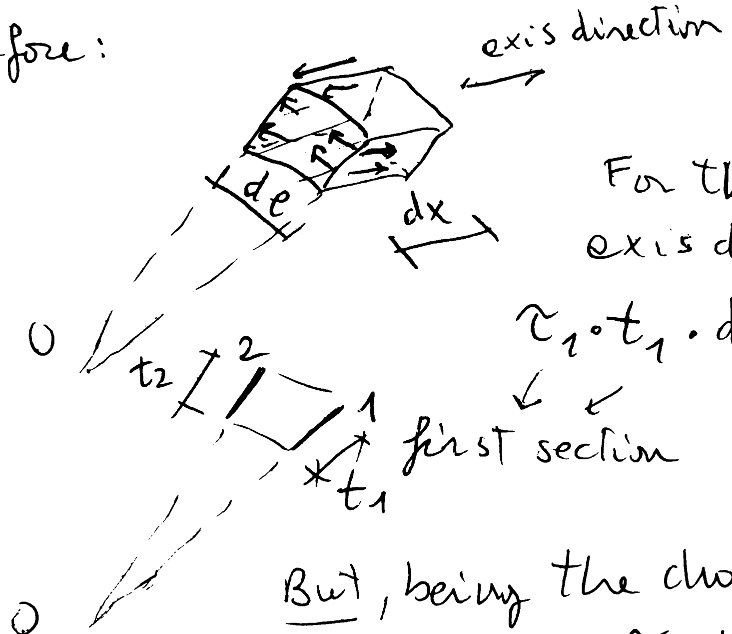
The contribution of dF, which can be considered applied at the barycentre of dA, to the torsional moment is $dT = dF \cdot R$ lever arm



Now, if we consider that portion extending for a distance dx along the longitudinal axis of the beam, we can derive an important result, if we recall a property of tangential stresses: if you consider an infinitesimal element, due to the rotational equilibrium, tangential stresses acting on orthogonal surfaces ~~no~~ run towards or away the same edge :



Therefore :



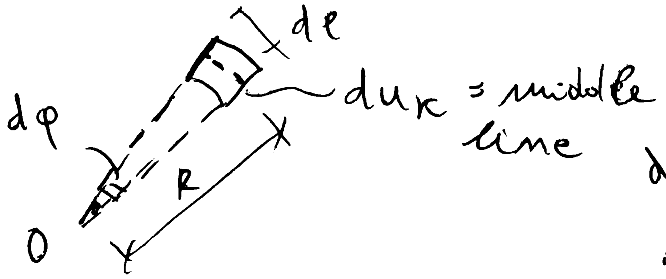
For the equilibrium along the axis direction, we have that

$$\tau_1 \cdot t_1 \cdot dx = \tau_2 \cdot t_2 \cdot dx \Rightarrow \tau_1 t_1 = \tau_2 t_2$$

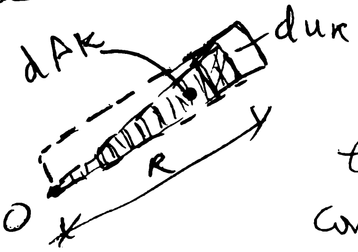
first section — second section

But, being the choice arbitrary, we derive that in every point $\tau \cdot t = \text{constant}$

Let's call that constant $q = \tau \cdot t$, and go back to the definition of the infinitesimal contribution to the torsional moment $dT = dF \cdot R$



$R \cdot d\ell$ = area of a rectangle



if we call dA_K the area of the ~~triangle~~ triangular slice comprised between O and $d\ell$, we get that

$$dA_K = \frac{1}{2} R \cdot d\ell$$

Thus, we can rewrite R as $2\, dA_K / d\ell$

We can then ~~refer~~ rewrite $dT = dF \cdot R = \int_{dA} \tau \cdot t \cdot d\ell \cdot \frac{2\, dA_K}{d\ell}$

moreover, $\tau \cdot t = \text{const} = q \Rightarrow dT = 2q \int_{dA} dA_K$

To get the whole torque T, we just need to integrate dT along the perimeter u_K $\longrightarrow$ $T = \oint_{u_K} dT = 2q \cdot A_K$

whole area enclosed by the perimeter u_K

Thus $q = \dfrac{T}{2 A_K}$

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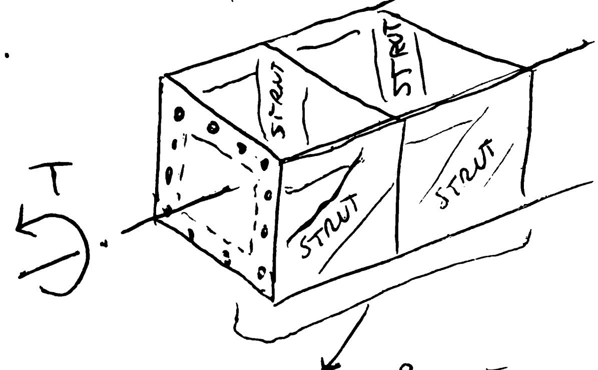
we can get this result, $q = \dfrac{T}{2A_k}$, and use it for our effective ideal hollow section, obtaining that $q = \tau_i \cdot t_{ef,i} = \dfrac{T_{Ed}}{2A_k}$ → design value of torque

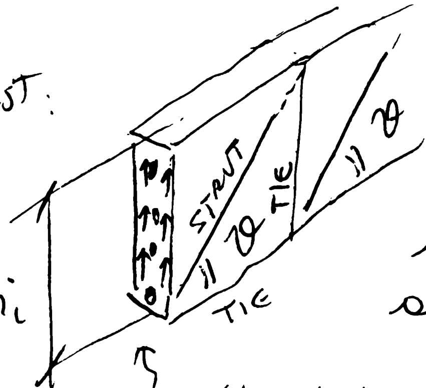
The effect of torsion and shear may be superimposed assuming the same value of ϑ, where ϑ is the angle of the (compressed) concrete strut used in the verification against shear ($1 \leq \cot\vartheta \leq 2.5 \Rightarrow 22° \leq \vartheta \leq 45°$)

From now onward, we consider just a rectangular cross section.



Each wall of the ideal hollow effective section can be studied separately

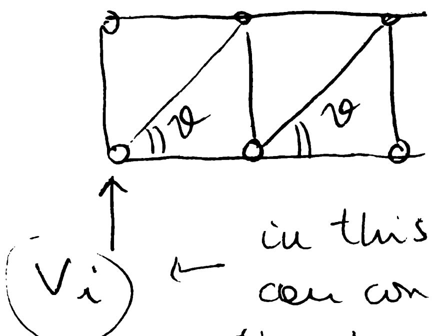
Let's consider this side first:



This wall can be modelled by a plane analogous truss

the integration of tangential stresses along this surface gives the portion of shear V_i that is withstood by this wall

plane analogous truss



in this model, we can consider that the shear force is applied to the node (actually, it would be spread along the surface)

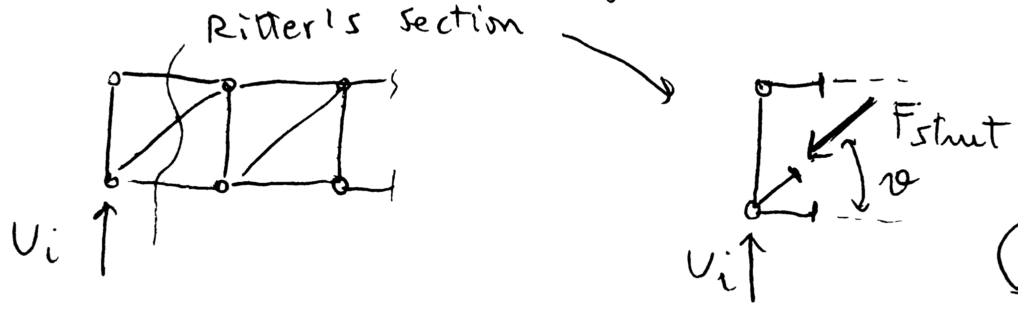
$$V_i = \int_{A_i} \tau \, dA$$

$$V_i = \int_{h_i} \underbrace{\tau \cdot t_i}_{\text{constant!}} \, dh$$

$$\boxed{V_i = q\, h_i}$$

$[\, q = T_{Ed}/2A_k \,]$

Now, we can derive the force in the inclined concrete strut in relation to the acting torque T_{Ed}



Equilibrium in vertical direction:

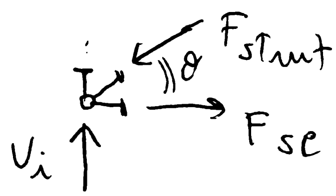
$$V_i = F_{strut} \cdot \sin\vartheta$$

$$F_{strut} = V_i / \sin\vartheta$$

$$F_{strut} = \frac{q\, h_i}{\sin\vartheta} = \frac{T_{Ed}\, h_i}{2 A_k \sin\vartheta}$$

- On the basis of the force in the strut, we can calculate also the force in the longitudinal reinforcement

Equilibrium of the bottom node: (horizontal direction)



$$F_{sl} = F_{strut} \cdot \cos\vartheta$$

$$\Rightarrow F_{sl} = \frac{T_{Ed} \cdot h_i \cos\vartheta}{2 A_k \sin\vartheta}$$

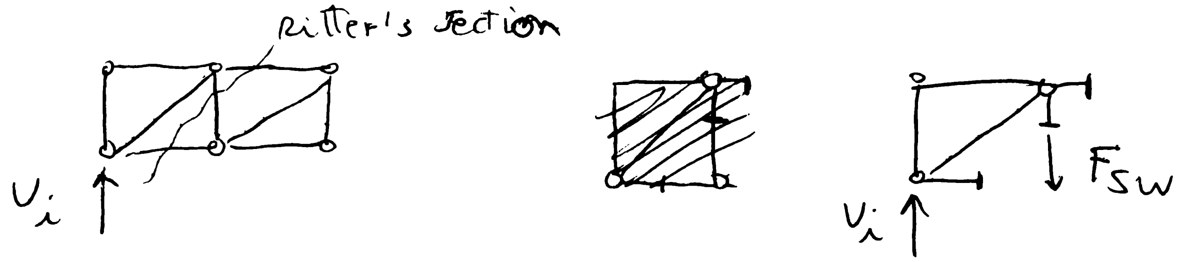
to get the overall longitudinal steel required, we have just to integrate (to sum) the values of F_{sl}

$$\sum F_{sl} = \sum \frac{T_{Ed}\, h_i}{2 A_k} \cot\vartheta = \frac{T_{Ed}}{2 A_k} \cot\vartheta \left(\sum h_i\right)$$

this is u_k

$$= \frac{T_{Ed}\, u_k}{2 A_k} \cot\vartheta$$

- Then, we need to calculate the vertical reinforcement



Equilibrium in the vertical direction:

$$F_{sw} = V_i = \frac{T_{Ed}\, h_i}{2 A_k}$$

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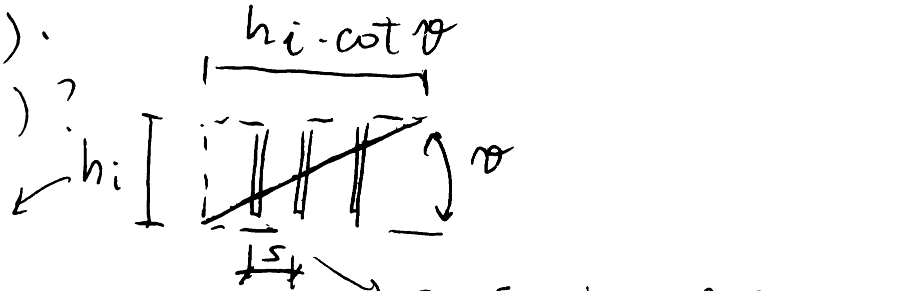
Of course, the force F_{sw} for the vertical steel (stirrups) is ~~suff~~ referred to n stirrups: more precisely, we assume that the force is carried by the stirrups ~~included~~ crossed by an inclined crack (parallel to the concrete struts), and is uniformly distributed among the n stirrups.

Pay attention that we are referring to a wall of the effective hollow section, then the area of steel related to a stirrup refers to a <u>single leg</u> of the stirrup (not to both, as in the case of shear, where you count twice the cross sectional area).

How many stirrups (n)?



$$n = \frac{h_i \cdot \cot\vartheta}{s} \rightarrow F_{sw} = \frac{1}{n} V_i = \frac{T_{Ed} \cdot s}{2 A_k \cot\vartheta}$$

for one stirrup

- What is the maximum torsional moment $T_{Rd,max}$ that the beam can withstand without the crushing of the concrete strut?

We have to assume a uniform pressure on the strut, and to equate that compression to a maximum value σ_{max}. This limit, σ_m, is derived by the design compressive strength of concrete, f_{cd}, but is decreased to account for other deleterious effects like a not-centred compression on the strut,

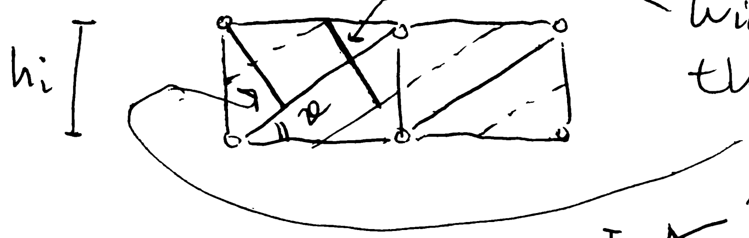
$$\sigma_{max} = \nu \cdot \alpha_{cw} \cdot f_{cd} \qquad \nu = 0.6\left(1 - \frac{f_{ck}}{250}\right)$$

f_{cd}: design compressive strength

$\alpha_{cw} = 1$ for nonprestressed concrete

f_{ck}: characteristic compressive strength of concrete, in MPa

To get $F_{strut,max}$ we need to calculate the area of the strut



width of the strut (the thickness is t_{ef})

⇓

$h_i \cdot \cos\vartheta$

Area of the strut: $\boxed{h_i \cos\vartheta \cdot t_{ef}}$

max force in the strut : $F_{strut,max} = \sigma_{max} \cdot$ Area of the strut

$$F_{strut,max} = \nu\, \alpha_{cw}\, f_{cd} \cdot h_i\, t_{ef} \cdot \cos\vartheta$$

Recalling that $F_{strut} = \dfrac{V_i}{\sin\vartheta} = \dfrac{T_{Ed}\, h_i}{2 A_k \sin\vartheta}$

$$\frac{T_{Rd,max} \cdot \cancel{h_i}}{2 A_k \cdot \sin\vartheta} = \nu\, \alpha_{cw}\, f_{cd} \cdot \cancel{h_i} \cdot t_{ef} \cdot \cos\vartheta$$

$$T_{Rd,max} = 2 \nu\, \alpha_{cw}\, f_{cd}\, A_k\, t_{ef} \cdot \sin\vartheta \cos\vartheta$$

- To evaluate the required amount of longitudinal steel, we can recall the relation to T_{Ed} (design torsional moment) and consider that the steel yields at the ultimate state.

$$\Sigma F_{se} = \Sigma A_{se} \cdot f_{yd} = \frac{T_{Ed}\, u_k}{2 A_k} \cot\vartheta \rightarrow \Sigma A_{se} = \frac{T_{Ed}\, u_k \cot\vartheta}{2 A_k f_{yd}}$$

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- Similarly, to evaluate the quantity of vertical stirrups distributed along the axial direction, we can recall the relation between the force and the torsional moment, and assuming that the steel yields at the ultimate limit state.

per unit length in the longitudinal direction

$$\frac{F_{sw}}{s} = \frac{T_{Ed}}{2A_k \cot\vartheta}$$

pay attention that this area refers to a single leg of the stirrup.

$$\frac{A_{sw} \cdot f_{yd}}{s} = \frac{T_{Ed}}{2A_k \cot\vartheta} \longrightarrow \frac{A_{sw}}{s} = \frac{T_{Ed}}{2A_k f_{yd} \cdot \cot\vartheta}$$

- The additional steel (longitudinal and vertical) required by torsion, and calculated as shown, on the basis of the design value of torsional moment T_{Ed}, has to be added to the reinforcement required by bending and shear design.

This addition can be realized by either increasing the diameter of longitudinal bars or by adding further bars that should be spread along the inner periphery of the links. In the case of smaller section, the bars could be concentrated at the corners.
Concerning the torsional links (stirrups), also in this case they should be added (i.e. the spacing, required by shear design, reduced). It is fundamental that those stirrups be closed, even welded in certain cases, and, however, not more spaced than $u_k/8$ apart.

- INTERACTION WITH SHEAR

The contemporary action of torsion and shear has to be evaluated by means of a simple interaction formula

$$\frac{T_{Ed}}{T_{Rd,max}} + \frac{V_{Ed}}{V_{Rd,max}} \leq 1.0$$

T_{Ed} and V_{Ed} are the design values of torsional moment and shear force, while $T_{Rd,max}$ is calculated as already seen (on the basis of the strength of the inclined concrete strut) and $V_{Rd,max}$ is calculated as follows:

$$V_{Rd,max} = \begin{cases} \alpha_{cw} \cdot b_w \cdot z \cdot \nu_1 \cdot f_{cd} \,/\, (\cot\vartheta + \tan\vartheta) & \text{for vertical shear reinforcements (stirrups)} \\ \alpha_{cw}\, b_w\, z\, \nu_1\, f_{cd}\, (\cot\vartheta + \cot\alpha) \,/\, (1+\cot^2\vartheta) & \text{for inclined shear reinforce (bent-up bars)} \end{cases}$$

$\alpha_{cw} = 1$ for ordinary r.c. (not pre-stressed)

b_w = width of the beam at the tension side (rectangular section $\rightarrow$ $b_w = b$, it could be different in the case of T-shaped beams wit sagging moments, for example)

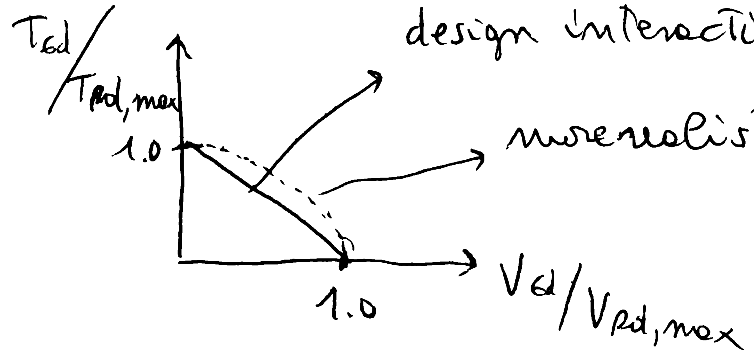
$\nu_1 = \nu$ (recommended) $\longrightarrow$ $\nu_1 = 0.6\left(1 - \frac{f_{ck}}{250}\right) = \nu$

z = lever arm of internal forces $\approx 0.9\,d$ $\rightarrow$ effective depth

ϑ = inclination of the concrete strut

α = inclination of the steel reinforcement ($\alpha = 90° \rightarrow$ stirrups)

$$T_{Rd,max} = 2\,\nu\,\alpha_{cw}\,f_{cd}\,A_k \cdot t_{ef} \cdot \sin\vartheta\,\cos\vartheta$$



design interaction diagram (conservative!)

more realistic interaction curve

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- In the case of rectangular (approximately) cross section, not heavily stressed by high values of torque and shear, you can avoid an evaluation of the additional reinforcement (just the minimum required is enough) provided that:

$$\frac{T_{Ed}}{T_{Rd,c}} + \frac{V_{Ed}}{V_{Rd,c}} \leq 1.0$$

$T_{Rd,c}$ is the torsional moment that produces cracking, it's calculated assuming $\tau = f_{ctd}$

$V_{Rd,c}$ is the shear resistence of members without shear reinforcement

$$T = \oint_{u_K} dT = 2q A_K = 2\overset{f_{ctd}}{\tau} \cdot t_{ef} A_K \Rightarrow T_{Rd,c} = 2 f_{ctd} \cdot t_{ef} A_K$$

$$V_{Rd,c} = \left[C_{Rd,c} \cdot k (100 \rho_\ell f_{ck})^{1/3} + k_1 \sigma_{cp} \right] b_w \cdot d$$

with a minimum value $V_{Rd,c,min} = (v_{min} + k_1 \sigma_{cp}) b_w d$

$$C_{Rd,c} = \frac{0.18}{\gamma_c} \qquad v_{min} = 0.035\, k^{3/2} \cdot f_{ck}^{1/2}$$

$$k_1 = 0.15 \qquad k = 1 + \sqrt{\frac{200}{d}} \leq 2.0 \quad (d \text{ in mm})$$

$$\rho_\ell = \frac{A_{s\ell}}{b_w d} \leq 0.02$$

$A_{s\ell}$ = area of tensile reinforcement, which extends more than $(\ell_{bd} + d)$ beyond the section considered

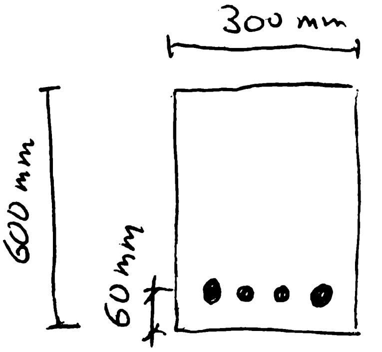
b_w = smallest width of the cross section

$$\sigma_{cp} = \frac{N_{Ed}}{A_c} < 0.2 f_{cd} \ (\text{MPa})$$

$N_{Ed} \rightarrow$ axial force due to loading or prestressing

$A_c \rightarrow$ area of concrete

$\boxed{\text{EXAMPLE}}$ DESIGN AGAINST TORSION



Beam already designed in bending and shear

Longitudinal steel for bending : $2\phi 32 + 2\phi 25$

Torsion : $T_{Ed} = 24$ kN·m

Shear : $V_{Ed} = 280$ kN.

(stirrups) → Vertical steel required by shear : $475\ mm^2/m$

Strut inclination $\vartheta = 22°$

Concrete C30/37 , steel $f_{yk} = 500$ MPa

Step 1 Design properties $f_{cd} = \dfrac{f_{ck}}{\gamma_c} \cdot \alpha = \dfrac{0.85 \cdot 30}{1.5} = 17$ MPa

$f_{ctd} = \dfrac{1}{1.5} \cdot 0.3\, f_{ck}^{2/3} = \dfrac{2.90}{1.5} = 1.93$ MPa $\quad f_{yd} = \dfrac{f_{yk}}{1.15} = 435$ MPa

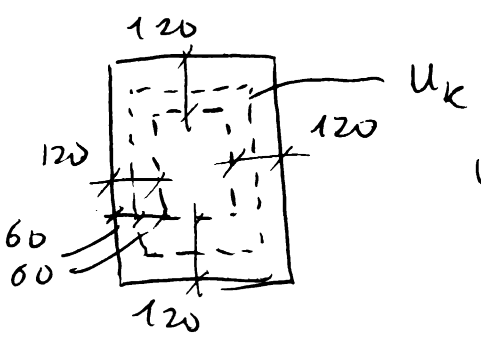
Step 2 Find the equivalent / effective hollow section

$$t_{ef} = \frac{A}{u} = \frac{300 \times 600}{2 \times (600 + 300)} = 100 \text{ mm}$$

minimum t_{ef} = 2 times the distance from the centreline of the longitudinal reinforcement from the external side → $t_{ef,min} = 2 \times 60$ mm $= 120$ mm

↓

$t_{ef} = 120$ mm



$u_k = 2 \times (300 - 120 + 600 - 120)$

$= 2 \times (180 + 480)$

$= 1320$ mm

$A_k = 180 \times 480 = 86400$

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Step 3 check if the section can withstand this combination of torque and shear

$$T_{Rd,max} = 2\nu\,\alpha_{cw}\,f_{cd}\,A_k\,t_{ef}\,\sin\vartheta\cos\vartheta \longrightarrow \vartheta = 22°$$

$\alpha_{cw} \to 1$

$$\nu = 0.6\left(1-\frac{f_{ck}}{250}\right) = 0.6\left(1-\frac{30}{250}\right) = 0.528$$

$$= 2\cdot 0.528\cdot 1\cdot 17\cdot 8.64\cdot 10^4\cdot 120\cdot 0.375\cdot 0.927$$

$$= 64.7\cdot 10^6\ N\cdot mm = 64.7\ kN\cdot m$$

$$V_{Rd,max} = \alpha_{cw}\,b_w\,z\,\nu_1\,f_{cd}/(\cot\vartheta + \tan\vartheta)$$

$\alpha_{cw} \to 1$, $b_w \to b$, $z \to 0.9d$, $\nu_1 \to \nu$

$$= 1\cdot 300\cdot 0.9\cdot 540\cdot 0.528\cdot 17/(2.475+0.404)$$

$$= 454.6\cdot 10^3\ N = 454.6\ kN$$

$$\frac{T_{Ed}}{T_{Rd,max}} + \frac{V_{Ed}}{V_{Rd,max}} \le 1 \longrightarrow \frac{24}{64.7} + \frac{280}{454.6} = 0.987 \le 1$$

Being so high the design forces, probably the other interaction formula $\left(\frac{T_{Ed}}{T_{Rd,c}} + \frac{V_{Ed}}{V_{Rd,c}} \le 1\right)$ would not be satisfied

Let's check it:

$$T_{Rd,c} = [\tau = f_{ctd}] = 2 f_{ctd}\cdot t_{ef}\cdot A_k$$

$$= 2\cdot 1.93\cdot 120\cdot 86400 = 40.0\ kN\cdot m$$

$$V_{Rd,c} = \left[C_{Rd,c}\cdot k\,(100\,\rho_\ell\cdot f_{ck})^{1/3} + k_1\,\sigma_{cp}\right] b_w d \qquad \sigma_{cp} = 0$$

$$C_{Rd,c} = \frac{0.18}{1.5} = 0.12 \qquad k = 1+\sqrt{\frac{200}{d}} = 1.61 \le 2.0$$

$$\rho_\ell = \frac{A_{s\ell}}{b_w d} = 0.016 \le 0.02$$

$$\to V_{Rd,c} = \left(0.12\cdot 1.61\,(100\cdot 0.016\cdot 30)^{1/3}\right) 300\cdot 540 = 113.7\ kN$$

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$$\frac{24}{40} + \frac{280}{113.7} > 1 \quad !$$

Step 4 Let's calculate the amount of vertical reinforcement required by torsion

$$\frac{A_{sw}}{s} = \frac{\overset{\to 24 \cdot 10^6 \ Nmm}{T_{Ed}}}{2 \underset{86.4 \cdot 10^3}{A_k} \underset{435}{f_{yd}} \underset{2.475}{\cot\vartheta}} = 0.129 \ mm^2/mm = 129 \ mm^2/m$$

→ area referred to one leg of the stirrups.

Overall amount of vertical reinforcement: $(A_{sw})_{shear} + (A_{sw})_{torsion}$

$(A_{sw})_{torsion}$ → for both legs; referred to two legs!

$$\underbrace{2 \cdot 129 \ mm^2/2}_{(A_{sw})_{torsion}} + 475 \ mm^2/m = 733 \ mm^2/m$$

Using stirrups $\phi 8$ → Area $= \underbrace{50.26}_{1 \ leg} \to 100.52$ (2 legs)

$$\frac{\text{Required area}}{\text{area of 1 stirrup}} = \text{number of required stirrups} \qquad \frac{733}{100.5} = 7.29 \text{ stirrups/metre}$$

$\longrightarrow \boxed{8 \text{ stirrups } \phi 8/m}$ (spacing $\frac{1m}{8} = 12.5$ cm)

minimum spacing $s_{min} < \frac{u_k}{8} = \frac{1320}{8} = 16.5 \text{ cm} > 12.5$! OK

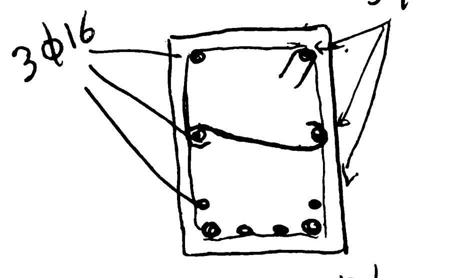
Step 5 Calculation of the additional longitudinal reinforcement

$$\Sigma A_{sl} = \frac{T_{Ed} \cdot u_k \cot\vartheta}{2 A_k \cdot f_{yd}} = \frac{24 \cdot 10^6 \cdot 1320 \cdot 2.475}{2 \cdot 86400 \cdot 435} = 1043 \ mm^2$$

$1 \phi 16 \to 201 \ mm^2 \to 1043/201 = 5.18 \to 6 \phi 16$

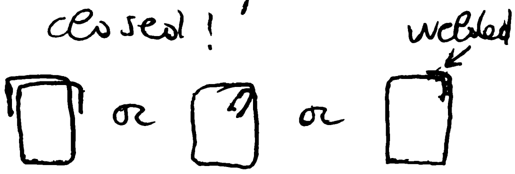
$1 \phi 20 \to 314 \ mm^2 \to 1043/314 = 3.3 \to 4 \phi 20$

$1 \phi 24 \to 491 \ mm^2 \to 1043/491 = 2.12 \to 3 \phi 24$



Suitable choice: $\boxed{6 \ \phi 16}$ longitudinal bars, which can be placed at the corners and in the middle of the longer edge

Stirrups must be closed!



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